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1903.

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PREPARED UNDER THE DIRECTION OF

COLONEL ST. G. C. GORE, C.S.I., R.E.,

SURVEYOR GENERAL OF INDIA.

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I. Longitude employed in the maps of the Survey of India.
II. Levelling across the Ganges.
III. Changes in a levelling staff due to moisture and temperature.
IV. A new form of sun-dial.
V. Nickel-steel alloys.
VI. Theory of electric projectors.

PREPARED UNDER THE DIRECTION OF

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I

ON THE VALUES OF LONGITUDE

EMPLOYED IN

MAPS OF THE SURVEY OF INDIA

BY

MAJOR S. G. BURRARD, R.E.,
SUPERINTENDENT TRIGONOMETRICAL SURVEYS.

ASTRONOMICAL DETERMINATIONS OF LONGITUDE MADE BETWEEN 1798 AND 1894.

The following determinations of the longitude of Madras were made prior to 1894:—

Date.	Longitude of Madras. Arc.			Longitude of Madras. Time.			Authority.	Book of Reference.
	°	′	″	h.	m.	s.		
1798	80	16	30	5	21	6·0	Lambton . . .	Asiatic Researches, Vol. X.
1805	80	18	30	5	21	14·0	Lambton . . .	Asiatic Researches, Vol. XII.
1815	80	17	21	5	21	9·4	Warren . . .	G. T. Survey of India, Vol. II.
1826	80	17	15	5	21	9·0	Goldingham . .	Records of the Madras Observatory.
1831	80	15	55·5	5	21	3·7	Taylor . . .	Vol. XVI, Memoirs, R. A. S.
1840	80	13	55·5	5	20	55·7	Riddle . . .	Vol. XII, Memoirs, R. A. S.
1845	80	14	19·2	5	20	57·28	Taylor . . .	Vol. XVI, Memoirs, R. A. S.
1847	80	15	56·55	5	21	3·77	Everest . . .	Meridional Arc of India.
1858	80	14	19·5	5	20	57·3	Jacob	G. T. Survey of India, Vol. II.
1878	80	14	51·24	5	20	59·416	Campbell . . .	Annual Report on the Great Trigonometrical Survey of India for 1876-77.
1883	80	14	50·03	5	20	59·335	Walker . . .	G. T. Survey of India, Vol. IX.
1890	80	14	51·08	5	20	59·405	Strahan . . .	G. T. Survey of India, Vol. XV.
1893	80	14	51·33	5	20	59·422	Strahan . . .	G. T. Survey of India, Vol. XV.

The last four values are different discussions of the same observations.

No account exists of Lambton's method of observation: his results alone remain on record. Warren's and Goldingham's values were deduced from the observations of Jupiter's satellites, and Taylor's from moon culminations. Riddle's, Everest's and Jacob's values were obtained from discussions of Taylor's lunar observations.

In 1874 the telegraphic longitude of Suez was measured by members of the Transit of Venus Expedition, and two years later the difference of longitude between Suez and Madras was telegraphically determined by Colonel (now General) Campbell and Colonel Heaviside. The combined result of these two operations was to place the Madras Observatory in longitude $5^h\ 20^m\ 59^s{\cdot}416$, a value subsequently modified by General Walker to $5^h\ 20^m\ 59^s{\cdot}335$.

The next modification made was in 1889: in the fourteen years previous to this date, a net-work of longitude triangles had been gradually thrown over the Indian Peninsula, the accuracy of the arcs of each triangle being tested by the smallness of the closing error; as early as 1876 the closing errors of the triangular circuits were considered unsatisfactorily large, in 1881 they averaged a quarter of a second of time, and in 1885 they became so large that it was considered useless to proceed with the work, unless their cause were discovered.

In 1889 these errors were proved to be due to imperfections in the object glasses of the collimators, and to eliminate the effect of these imperfections a new method of calculating the collimation-constant was introduced: all the Indian arcs of longitude, including Bombay-Āden and Aden-Suez, had consequently to be computed *de novo;* the large circuit errors were then found to have disappeared, and the longitude of Madras became $5^h\ 20^m\ 59^s{\cdot}405$.

When all the Indian longitude arcs were finally adjusted in 1893, by a simultaneous reduction by the method of minimum squares, the difference of longitude between Bombay and Madras was increased by $0^s{\cdot}017$, and the longitude of Madras made $5^h\ 20^m\ 59^s{\cdot}422$.

THE LONGITUDE OF THE GREAT TRIGONOMETRICAL SURVEY OF INDIA.

No change has been made in the original value of longitude adopted for the Great Trigonometrical Survey of India; that value was Warren's value, 80° 17′ 21″, and was introduced by Colonel Lambton in 1815.

The precise error in the longitude of the principal triangulation is not, however, identical with the error in Warren's longitude of Madras: Kaliánpur is the origin of the triangulation and its longitude was fixed by Colonel Everest as follows:—

	°	′	″
Warren's longitude of Madras	80	17	21
Difference of longitude between Madras and Kaliánpur found by Triangulation	2	35	36·25
Everest's longitude of Kaliánpur	77	41	44·75

In Volume II of the *Account of the Operations of the Great Trigonometrical Survey of India*, General Walker contemplated the possibility of the quantity 2° 35′ 36″·25 having to be modified in future, when his revisionary triangulation between Kaliánpur and Madras had been finally reduced,* and when possibly Clarke's spheroid had been substituted for Everest's; but in 1884 it was pointed out by Mr. Hennessey that the difference of longitude between Kaliánpur and Madras should be determined astronomically.

* The revision of the triangulation gave the result 2° 35′ 36″·63: Everest's spheroid was employed both in the original and the revisionary calculations.

The astronomical determination was made in 1889, and the difference of longitude found to be 2° 35′ 29″·49. The actual error, therefore, in the adopted longitude of the Great Trigonometrical Survey of India is 6″·76 *less* than the error of Warren's Madras value.

In 1840 Everest estimated the error in longitude of the Indian triangulation at about + 3′ 30″ : in Volume II of the *Account of the Operations of the Great Trigonometrical Survey of India*, General Walker, using Jacob's value of Madras, gave it as about + 3′, but in the interim between the printing and publication of that volume the first telegraphic determination of longitude was carried out, and in the preface to the volume General Walker reduced his estimate to about + 2′ 30″, the difference between Campbell's and Warren's values In Volume XV of the *Account of the Operations of the Great Trigonometrical Survey of India*, Colonel G. Strahan calculated the error to be +2′ 22″·92.

Origin of the Longitude Operations undertaken in 1894-95-96.

In 1891, at a meeting of the International Geographical Congress at Berne, the question was raised as to why the Government of India did not correct the longitude of its maps, instead of continuing to publish them with an acknowledged error of 2′ 30″.

A discussion followed in India as to whether any such alteration was feasible, and upon examination the longitude of India was found even then to be not known with *sufficient accuracy* to justify a change. Colonel Everest had to deal with the same problem half a century before. The longitude of his triangulation had been made dependent on Warren's value for Madras, but as time went on, Taylor had improved on Warren's value, and Colonel Hodgson, the Surveyor General of India, had had other independent observations taken at Calcutta: Everest had to decide whether he would make use of these later observations at Madras and Calcutta and substitute a new value of longitude for Warren's. His decision had best be given in his own words: "These "data seem to me by no means *sufficiently* conclusive to warrant any alteration in the quantities "employed by Colonel Lambton in all previous operations of the Great Trigonometrical Survey, "for just in the same manner as Mr. Taylor has assigned a new value for the longitude of Madras, "some future astronomer may introduce another alteration. In fact the actual determination of "the terrestrial longitude of any place is too difficult and delicate a question to rest on a small "number of observations, and if every new set of determinations were appealed to as a test, there "would be no end to the shifting of the origin: wherefore it seems to me better, for the present "at least, to use the same value as that employed by Colonel Lambton."

In 1815 Captain Warren's value was the best attainable, but in 1877 it was rendered obsolete by the telegraphic determination *viâ* Mokattam and Suez. Unfortunately this latter measurement, superior as it is to all previous results, has itself been subjected to somewhat severe criticism, and though it is sufficiently accurate to prove conclusively the existence of considerable errors in Warren's and Taylor's values, it is held to have been by no means determined with the highest accuracy attainable. In Volume I of the Annals of the Cape Observatory, Sir David Gill, K.C.B., F.R.S., H. M.'s Astronomer at the Cape, discussing the longitude of the Cape of Good Hope, which like that of Madras depends on the telegraphic determinations between Greenwich, Mokattam, Suez and Aden, writes: "The weak point of this result is unquestionably the determination "of the longitude Greenwich-Aden. Neither of the two series of operations on which it depends "was executed with such refinements or precautions as are necessary for the determination of "fundamental longitudes, nor indeed, so far as I know, were these operations planned with a view "to the securing of more accuracy than would amply suffice for Transit of Venus purposes.

"For refined purposes, the results of the British Transit of Venus party are vitiated by the "extraordinary variations of the Personal Equation of the observers engaged in the determination "of the Greenwich-Mokattam longitude, the results varying over a range of six-tenths of a second "of time on the seven nights upon which Personal Equation was determined.

"For the longitude Mokattam-Suez there is no comparison of the Personal Equation of the "observers before the expedition, and only a somewhat unsatisfactory one after it."

In 1893, when the reliability of the Mokattam-Suez-Aden arcs was under discussion, an astronomical party was ordered to proceed to Baluchistan and Persia to determine telegraphically the longitudes of points on the Makrán Coast and in the Persian Gulf. An opportunity was thus presented of obtaining *viâ* Tehran a new and refined determination of the longitude of Madras.

The opportunity was taken, and the party was ordered to extend their operations through Persia and Europe to Greenwich.

RESULTS OF THE OBSERVATIONS TAKEN IN 1894-95-96.

Six arcs of longitude were observed between November 1894 and April 1896, the resulting values being as follows:—

Arc.	Observed Difference of Longitude.			Probable Error.
	h	*m*	*s*	*s*
Karachi-Jask	0	36	59·697	± 0·0057
Jask-Bushire	0	27	45·057	± 0·0095
Karachi-Bushire	1	4	44·812	± 0·0097
Tehran-Bushire	0	2	21·443	± 0·0083
Tehran-Potsdam	2	33	24·228	± 0·0068
Potsdam-Greenwich	0	52	15·953	± 0·0058

The first three of these arcs form a circuit, and its closing error may be deduced thus:—

	h	*m*	*s*
Karachi-Jask	0	36	59·697
Jask-Bushire	0	27	45·057
Sum .	1	4	44·754
Karachi-Bushire	1	4	44·812
Closing error .			0·058

The value to be adopted for the difference of longitude between Karachi and Bushire is obtained as follows:—

	h	*m*	*s*	*s*
Karachi-Jask	0	36	59·697	± 0·0057
Jask-Bushire	0	27	45·057	± 0·0095
Karachi-Bushire (indirect)	1	4	44·754	± 0·0110
Karachi-Bushire (direct)	1	4	44·812	± 0·0097
Mean value, regard being paid to weights	1	4	44·787	± 0·0073

The longitude of Madras can now be calculated, the values of the three arcs connecting Madras and Karachi being taken from pages 440 and 441 of Volume XV of the *Account of the Operations of the Great Trigonometrical Survey of India.*

Arc.	Difference of Longitude.	Probable Error.	Station.	Longitude East of Greenwich.		Probable Error.	
				In Time.	In Arc.	In Time.	In Arc.
	h m s	s		h m s	° ′ ″	s	″
Potsdam-Greenwich	0 52 15·953	± 0·0058	Postdam .	0 52 15·953	13 3 59·30	± 0·0058	± 0·087
Tehran-Potsdam .	2 33 24·228	± 0·0068	Tehran .	3 25 40·181	51 25 2·72	± 0·0089	± 0·134
Tehran-Bushire .	0 2 21·443	± 0·0083	Bushire .	3 23 18·738	50 49 41·07	± 0·0122	± 0·183
Karachi-Bushire .	1 4 44·787	± 0·0073	Karachi .	4 28 3·525	67 0 52·88	± 0·0142	± 0·213
Bombay-Karachi .	0 23 12·196	± 0·0129	Bombay .	4 51 15·721	72 48 55·82	± 0·0192	± 0·288
Bolarum-Bombay .	0 22 48·801	± 0·0061	Bolarum .	5 14 4·522	78 31 7·83	± 0·0201	± 0·302
Madras-Bolarum .	0 6 54·615	± 0·0085	Madras .	5 20 59·137	80 14 47·06	± 0·0219	± 0·329

The error in the value of longitude adopted for the triangulation of the Great Trigonometrical Survey of India is thus +2′ 27″·18, being equal to (80° 17′ 21″—80° 14′ 47″·06—6″·76).

The Values of Longitudes employed in Indian Maps.

Prior to the year 1900 there had always been two values of longitude employed in the mapping of the Survey of India. The Atlas Sheets had been based on Lambton's value (1805) for the longitude of Madras Observatory, *viz.*, 80° 18′ 30″: the Standard Sheets and all other mapping had been brought into accordance with the Great Trigonometrical Survey of India and based on Warren's value (1815) for the longitude of Madras, *viz.*, 80° 17′ 21″. In 1878 General Campbell's and Colonel Heaviside's Electro-Telegraphic observations of the difference of longitude between Madras, Aden and Suez showed that Lambton's and Warren's values were too large, and placed Madras in longitude 80° 14′ 51″: the error in the longitude of the Great Trigonometrical Survey of India was therefore shown to be 2′ 30″. After the completion of the Campbell-Heaviside determination a footnote was added to the maps of the Survey of India: on the Atlas Sheets this note ran thus:—

All longitudes require a correction of — 1′ 9″ to reduce them to the origin of the Great Trigonometrical Survey, *viz.*, the Madras Observatory taken as 80° 17′ 21″ East of Greenwich, and a further correction of — 2′ 30″ to reduce them to the latest value 80° 14′ 51″ of that Observatory.

On the Standard Sheets and on all mapping based on the Great Trigonometrical Survey the footnote was as follows:—

The longitudes are referrible to the Greenwich Meridian, taking that of Madras Observatory as 80° 17′ 21″ East. They require a correction of — 2′ 30″ to make them accord with the most recent value of that observatory, *viz.*, 80° 14′ 51″.

In 1896 the observations of the difference of longitude between Karachi and Greenwich *viâ* Tehran and Potsdam were completed, and placed Madras in longitude 80° 14′ 47″·06, thus increasing the apparent error of the Great Trigonometrical Survey to 2′ 34″.*

In the year 1900 the construction of a Map of India and Adjacent Countries on the scale of $\frac{1}{1000000}$ was commenced, and the question arose as to whether its longitude should be made to accord with that of the Great Trigonometrical Survey, or with the most recent determination, or whether a value of longitude should be obtained by combining the result of the Perso European Arcs with that of the Aden-Suez Arcs, or with those of the German and American determinations. After full consideration it was decided by Colonel Gore, *firstly*, to give the Map of India and Adjacent Countries a new and correct value of longitude, instead of continuing the value adopted by the Great Trigonometrical Survey; and *secondly* to adopt for the new value the determination, made in 1894-96 by Captains Burrard and Lenox Conyngham, *viâ* Karachi, Tehran and Potsdam.

Another question, however, had to be decided: Kaliánpur is the origin of the Indian Survey: the Madras Observatory happened to be the place at which the earliest observations for longitude were taken: and the method which Colonel Everest employed of determining the longitude of Kaliánpur was to accept a value for the longitude of Madras and to calculate the differential longitude of Kaliánpur by means of the triangulation. The question that had now to be decided was: should the new value of longitude, which was to be introduced into Indian mapping, be made to accord with the astronomical value of Madras or with that of Kaliánpur?

The difference of longitude between Madras and Kaliánpur as determined astronomically is 7″·14 *less* than as measured by the revisionary triangulation. We do not know the cause of this discrepancy: the astronomical value may have been affected by Himalayan or continental or local attraction; the value deduced from the triangulation may have been rendered incorrect by the employment of Everest's values of the axes of the earth: it is not possible to say at present whether the astronomical or the geodetic determination is the more reliable.

If the astronomical value for the longitude of Madras were adopted, *viz.*, 80° 14′ 47″·06, as a basis for the map, the longitude of Kaliánpur calculated from the triangulation would be 77° 39′ 10″·43. If the astronomical value for the longitude of Kaliánpur were adopted as the basis, *viz.*, (80° 14′ 47″·06 — 2° 35′ 29″·49) = 77° 39′ 17″·57, the longitude of Madras calculated from the triangulation would be 80° 14′ 54″·20. The question at issue was not whether one alternative was more correct than the other, but which of two equally correct methods it was more advisable to adopt. By one method the Map of India would be placed 7″·14 nearer to the meridian of Greenwich than by the other.

Kaliánpur was made the astronomical origin of latitude by Colonel Everest in 1840; in 1900 Colonel Gore decided to make it the astronomical origin of longitude also: the longitude of Madras will henceforth be deduced from the triangulation. A reference to Madras has to be retained in the footnotes on maps, because it is the only point at which a comparison between recent and former values of longitude can be instituted: the addition, however, of the word "Geodetic" to the footnote shows that Madras has ceased to be the astronomical origin of longitude.

* I say "apparent" error, because 80° 14′ 47″ is 2′ 34″ less than 80° 17′ 21″: as the quoted footnote stated a correction of 2′ 30″ to be necessary, when the value 80° 14′ 51″ was believed correct, it would have been but reasonable to assume that the correction would have to be increased to 2′ 34″, when the value 80° 14′ 47″ came to be substituted. But the problem was complicated in the early years of the century by the introduction of *two* stations of origin, Kaliánpur and Madras; the triangulation of India was made to emanate from Kaliánpur, whose longitude was unknown, and it picked up a value of longitude, when it was connected with Madras. In the old footnotes to maps it was tacitly *assumed* that the astronomical value of the difference of longitude between Kaliánpur and Madras did not differ from the value derived from the triangulation: the astronomical value was, however, in reality 6″·76 less than the value given by the old triangulation and 7″·14 less than the value given by the revisionary triangulation. No cognisance was taken of this discrepancy in the old footnotes; the observations of 1894-95-96 showed the true error in longitude of the Great Trigonometrical Survey to be 2′ 27″; the "apparent" error 2′ 34″, deduced from a superficial consideration of the old footnotes, was 7″ too large.

In accordance with Colonel Gore's Departmental Order No. 13 of the 17th of May, 1900, the following footnote is now placed on sheets of the Map of India and Adjacent Countries on the the scale of $\frac{1}{1000000}$:—

The Longitudes are referrible to the Greenwich Meridian, taking that of Madras Observatory as 80° 14′ 54″ East, the most recent Geodetic value.

For newly-engraved Atlas Sheets the footnote will in future be as follows :—

The Longitudes are referrible to the Greenwich Meridian, taking that of Madras Observatory as 80° 18′ 30″ East. They require a correction of — 1′ 9″ to make them accord with the Great Trigonometrical Survey, and a further correction of — 2′ 27″ to make them accord with the most recent value of the Geodetic Longitude of Madras Observatory, *viz.*, 80° 14′ 54″.

For standard sheets and all mapping based on the longitude of the Great Trigonometrical Survey of India, the footnote will in future be as follows :—

The Longitudes are referrible to the Greenwich Meridian, taking that of Madras Observatory as 80° 17′ 21″ East. They require a correction of — 2′ 27″ to make them accord with the most recent value of the Geodetic Longitude of that Observatory, *viz.*, 80° 14′ 54″.

S. G. BURRARD.

II

LEVELLING ACROSS THE GANGES AT DAMUKDIA.

BY

CAPTAIN H. L. CROSTHWAIT, R.E.

The object of the experiments described in this paper was to determine the best way of carrying levels of precision across the large rivers, which are occasionally met with, in the levelling operations of the Survey of India.

In the field seasons of 1899-1900 and 1900-1901, a line of levels was carried along the Eastern Bengal State Railway from Calcutta to the Sona Khoda Base, and in the course of the work it became necessary to cross the Ganges near Damukdia. At this site the narrowest point, from bank to bank, where suitable foundations for the instruments could be found was about 102 chains, or a little over 1¼ miles.

On former occasions when a river had to be crossed, if it was too wide to allow the graduations of the staff to be read in the ordinary way, a position was selected where the stream was more or less symmetrical with regard to the two banks. Two graduated poles were driven in, one at each bank, to act as water gauges and their zeros connected by levelling one to a bench mark whose height was known and the other to one whose height was to be determined. Careful measurements were then made to determine the heights of the surface of the water on the two sides below these zeros and on the assumption that these heights were identical, the height of the second bench mark was determined. Some doubt having been cast on this method, the opportunity which presented itself at Damukdia was taken to conduct careful experiments with the object of testing whether the supposition that the heights of the surface water at two banks were identical is correct.

For this purpose it was decided that, in addition to this water gauge method, two independent ways of determining the difference of the height of bench-marks placed on opposite sides of the river should be used. The difference of heights of the two bench-marks was therefore determined *firstly*, by means of vertical angles, rigorously observed with two 24-inch theodolites on the system used in the Great Trigonometrical Survey, and *secondly*, by observations with two standarb levels to specially constructed discs. In each case two observers were employed, one on each side of the river, who took the observations simultaneously, so as to insure, as nearly as possible, the same atmospheric conditions.

The result of the operations was that the difference of heights between the bench-marks B and C shown in Fig. 1 was found to de:—

	No. of obs.	C above B. Feet.	P. E. Feet.
By vertical angles	72	2·139	± 0·005
By levelling	114	2·132	± 0·016
By water gauges	75	2·212	± 0·001

Selection of Site.

It was not at all easy to find a site suitable in every respect for the crossing. The channel of the river wanders from side to side, *i.e.*, the main current is in some places at one bank, and in some at the other. The state of affairs is illustrated in the diagram (Fig. 1).

The conditions sought were

(*a*) A symmetrical channel.

(*b*) As short as possible a distance between the banks.

At G H (Fig. 1) there was a more or less symmetrical channel at which suitable foundations could be found, as well as level ground for the back-sight necessary for levelling operations, but the distance 2·5 miles was much beyond the power of the levels. The crossing L M was for the same reason rejected. The line B C was finally chosen, where the distance from bank to bank was 1·28 miles, as determined from a measured base by triangulation.

The current was certainly towards the side C, but the velocity of the water did not exceed 2 miles an hour, and the river bed was quite straight for about ¾ of a mile above the crossing, and at right angles to B C. It seemed therefore probable that if the water was heaped up on the side C more than on the other, the difference would not be great, as might be the case at a bend in the river. Though there was a current, it was so slight, that when viewed from B on a calm day, the expanse of water appeared more like a great lake than a flowing river.

There was, however, one objection to the site chosen. The amount of dry sand was not the same on each side. On the side B a stretch of sand extended some 1,900 feet before the water's edge was reached, while at the other bank the water came up quite close to the instruments. This was no doubt unfortunate, but there was no better site available in the neighbourhood. During the day this stretch of sand was heated by the sun to a much greater extent than the water, so that a ray of light passing from B to C (Fig. 2) had to traverse a layer of atmosphere highly heated by contact with the hot sand extending from B to K, and then a comparatively cool layer over the water from K to C. As the distance K C was nearly three times that of K B, it is possible that the rays from B to C and those from C to B would not be equally affected by refraction, owing to the unsymmetrical position of K, the dividing line of sand and water with reference to B and C. That the effects of refraction may have been different for an object at C viewed from B, than for one at B viewed from C, was indicated though not proved by the fact that objects at C always appeared very much more unsteady, when seen from B, than those at B did when seen from C. It is not, however, easy to estimate the effect which this state of affairs would have on the results.

Instruments.

The instruments employed were:—

Two 24-inch theodolites, Nos. 1 and 2 by Barrow.

Two standard cylindrical levels, Nos. 3 and 4, of 21 inches focal length and 2½ inches object glass by Troughton and Simms.

Two 3-foot discs sliding on vertical uprights (see Fig. 5).

Two ordinary levelling staves, used for the back-sight in connection with the levels.

Two old levelling staves, bolted to piles driven in, one on each side of the river, to act as water gauges.

With the exception of these two old staves, all the instruments, including the discs and the first mentioned staves, rested on isolated masonry pillars, their general arrangement being shown in Fig. 4. The standard level stood on pillar I, the sliding disc on B, and the theodolite on E. The arrangements at both sides of the river were identical, except that the pillars I and B changed places as shown in Fig. 3, so as to bring the disc pillar always opposite its level. A and D in Fig. 3 are the two pillars onwhich the levelling staves used for the back-sights were placed when levelling. The distances and dimensions are given in Figs. 3 and 4. The two bench-marks B and C on either side of the river were taken as points of reference. The problem then resolved itself into finding, by the three methods already mentioned, the difference in level between them.

The following is a description of each method :—

By Vertical Angles.

Vertical angles were taken with the 24-inch theodolites, placed on pillars E and F. The signals observed to were the discs (Fig. 5) placed on B and C, and firmly clamped at a known height. The ordinary method of observing angles in first class work was followed and requires no special description Observations were conducted on three different days, always at the time of minimum refraction. They were taken simultaneously by two observers, one on each bank. Intersections were made almost at the same instant by each observer, by means of pre-arranged signals. Therefore atmospheric conditions were as nearly as possible the same. The distances E C and B F, though not strictly so, have each been taken equal to 1·28 miles, the error in length introduced being only a small fraction of an inch. The results are in very good accordance, giving a probable error of 0·005 of a foot.

By Levelling.

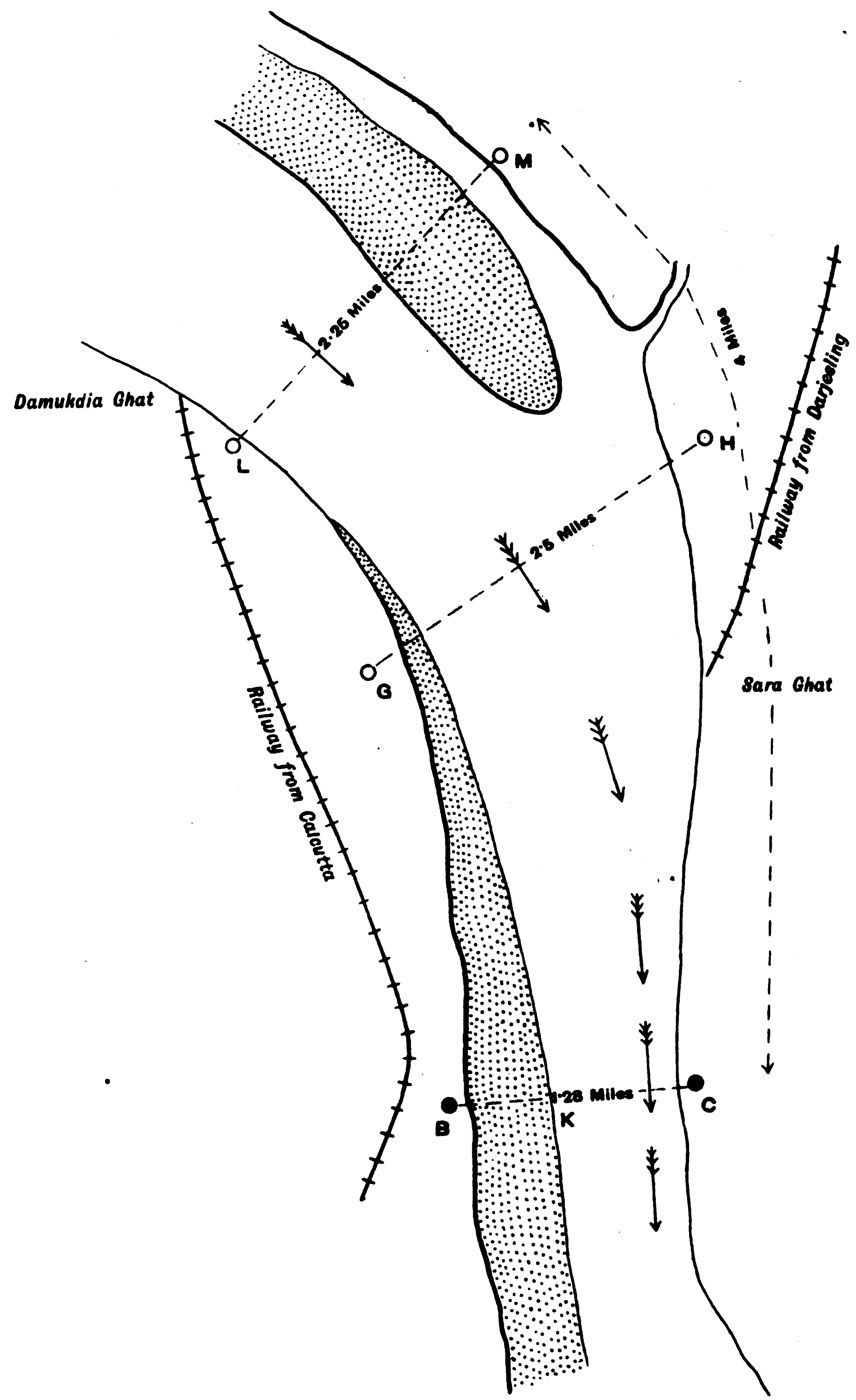
This method requires a somewhat longer description than the preceding one, which is well known in the Department. A description of operations as conducted from one side will suffice for both. The standard level was placed on a universal stand which rested on the pillar I (see Fig. 3). The framework which held the disc to be observed was erected and guyed on C, and the levelling staff on A. The bubble was brought as nearly as possible into the centre of its run, so that it remained practically stationary when the instrument was turned 180°. The object of this precaution was to keep the level correction as low as possible, which is important in this case, as any error in the determination of the value of a bubble division, would be much accentuated when dealing with such a long distance as 102 chains, nor would it tend to cancel on account of the great inequality of length in the fore-sight and back-sight. A reading was then taken on the staff placed on A, six chains distant, and recorded in column 6 of the attached specimen of field book. The object and eye ends of the bubble were read, the instrument turned through 180°, the bubble ends read again, and recorded, as shown in columns 14 and 15. The level correction with its proper sign, computed from these four readings, was used in correcting the staff reading. This method of reversing the level after each staff or disc reading was adhered to throughout, so that for every reading of height there were four bubble readings. Having read staff A, the telescope was turned on the disc at C, which was made to slide between the two guide pieces shown in Fig. 5, until it was well below the line of sight. It was then slowly raised until the white line painted on the disc was seen by the observer to be intersected by the horizontal wire of the level. The instant this was effected, a signal was made to the disc operator at C, who then clamped the disc by means of the thumb-screw shown at the back. The height of the centre of the disc above C was next carefully measured and recorded in column 11. The same operation was repeated, the disc being lowered to the intersecting position instead of being raised. A group of operations consisted of :—

Read A, the back-staff	Read C, disc falling	Read C, disc rising
„ C, disc rising	„ C, disc falling	„ A, the back-staff

The mean level correction for A from column 17 was then applied to the mean staff reading in column 6, and the mean for the disc from column 18 to the mean C reading in column 12, and the

Fig. 1

SKETCH-SITE OF OPERATIONS

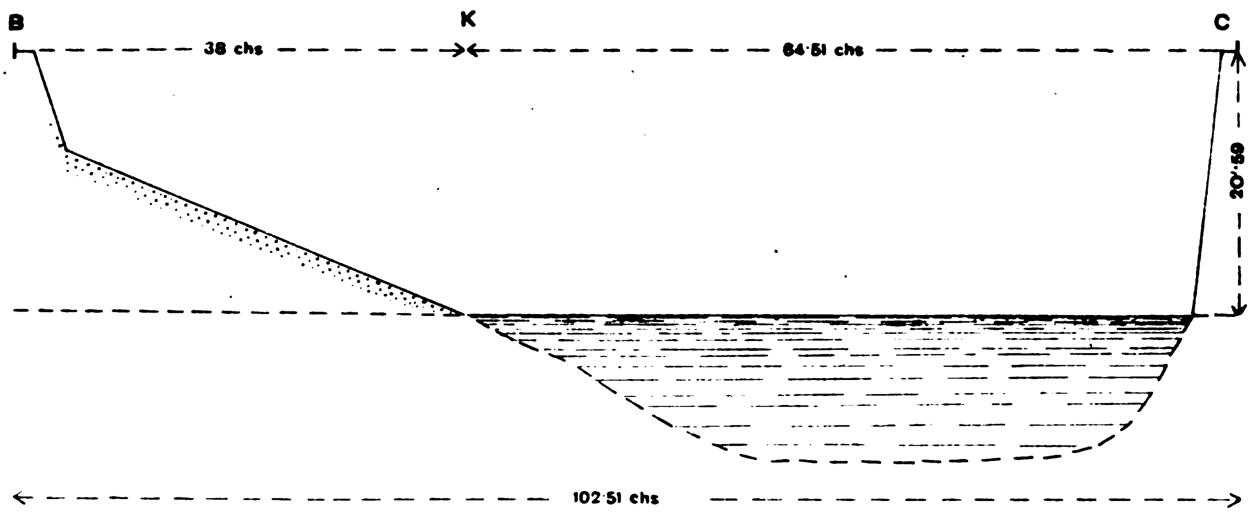


NOT TO SCALE

Fig. 2

CROSS SECTION

OF THE GANGES BETWEEN PILLARS A & B

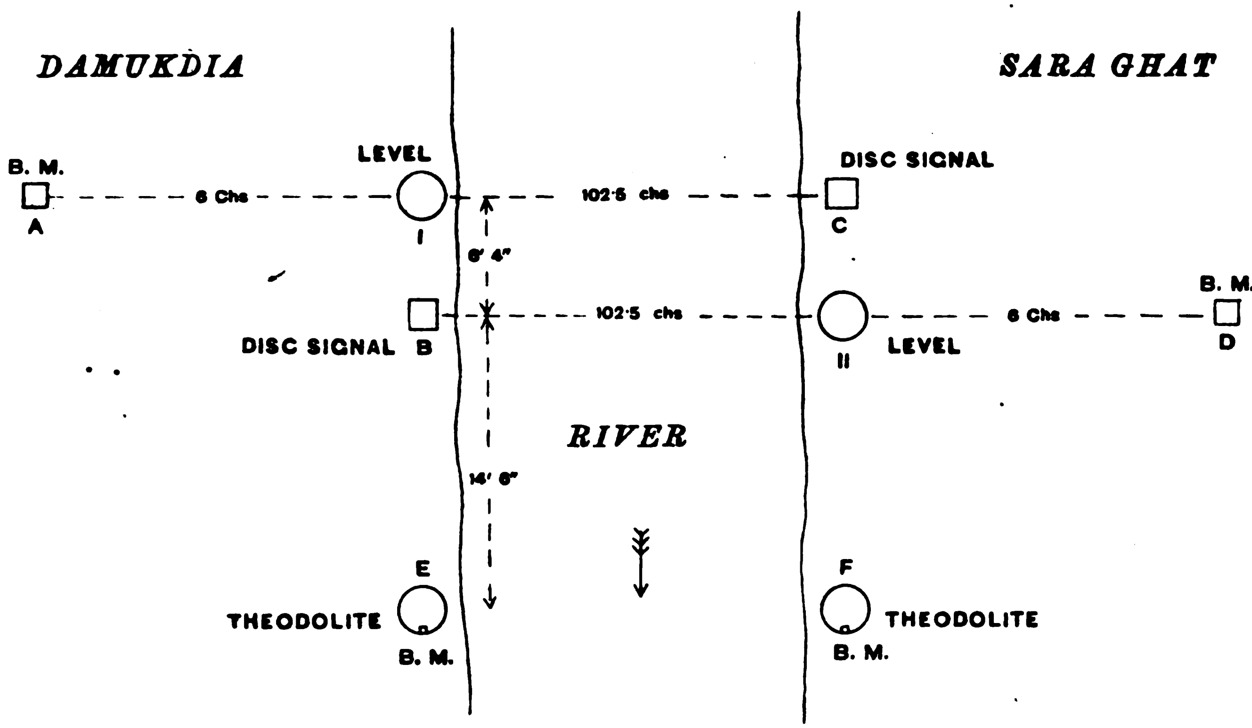


HORIZONTAL SCALE 1" = 1056′

VERTICAL SCALE 1" = 15′

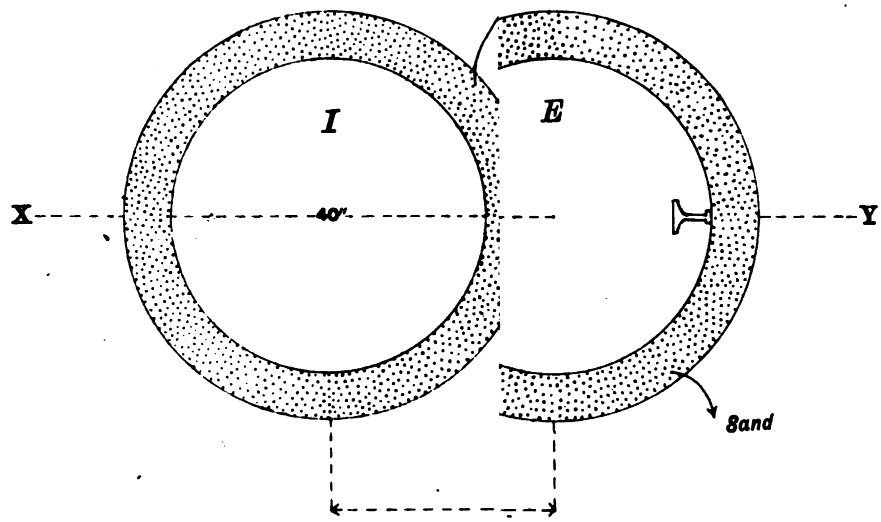
Fig. 3

DIAGRAM OF CROSSING PILLARS



NOT TO SCALE

Fig. 4



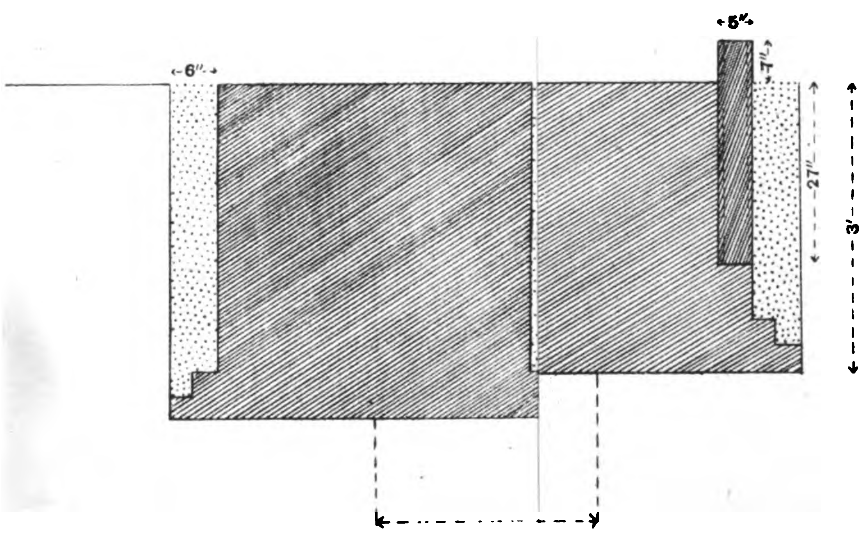
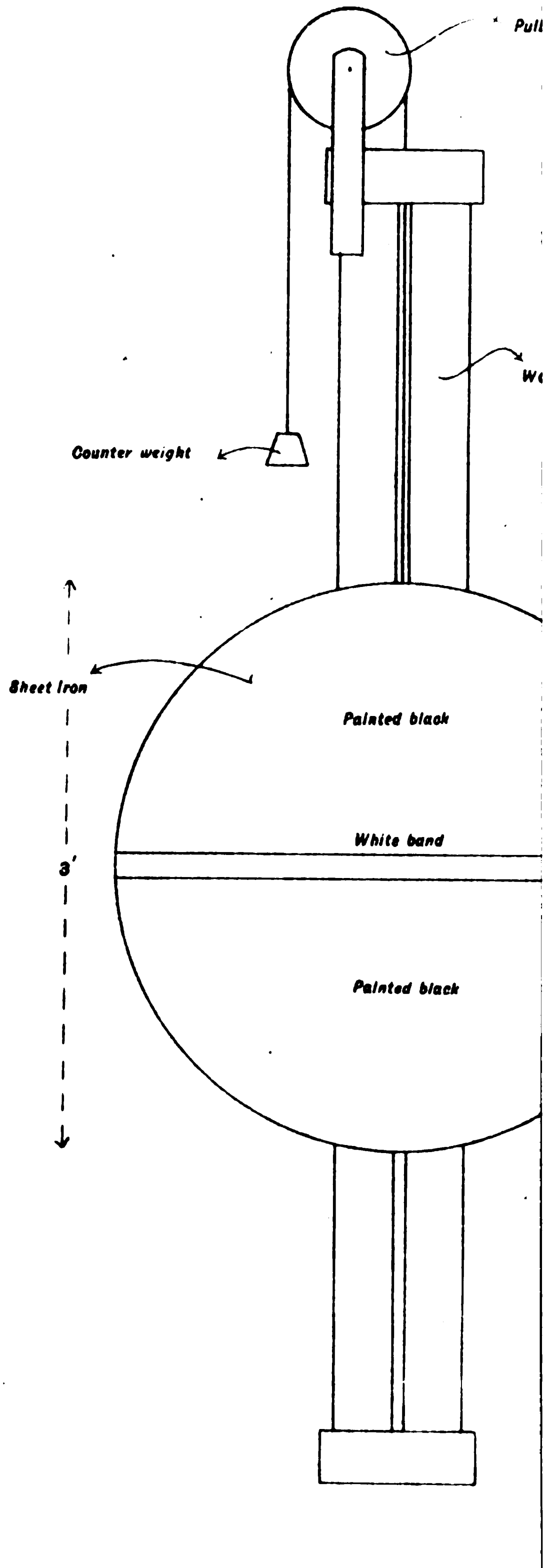
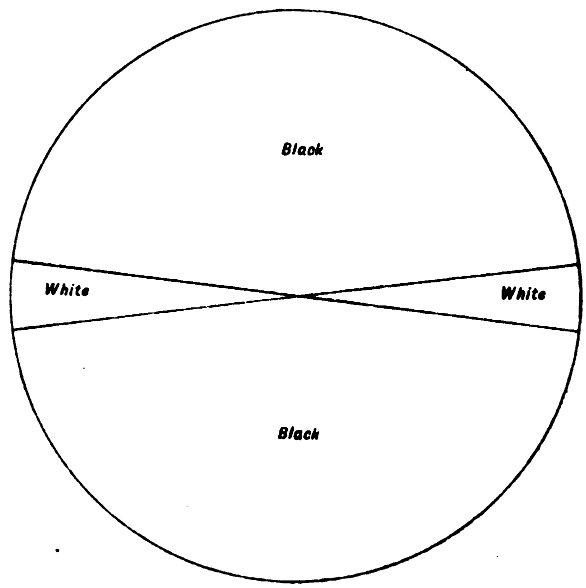


Fig. 5

FRONT ELEVATION



FINAL DISC



River Crossing — Ganges (Damukdia)

Level No. 4 — Observer... Lieut. H. L. Crosthwait R. E.

Value of 1 Division of Scale at 1 Chain = ·0005 — Disc Signal Operator.... E. H. Corridon at Saraghat (C)

Position of Level... Damukdia (I) (See diagram) — Novr. 28th 1900

(17) = Value of divn. × (16) × (5)
(18) " " " " × (16) × (9)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
			Staff				Disc Signal C						Bubble								
Date	Time	No. of Set	Position and No.	Distance	Reading	Corrected Reading	Position	Distance Chains	No. R / No. F	Recorded Reading	Mean R + F	Corrected Reading	Back End	Forward End	± ½ Difference	Correction to Staff Reading	Correction to Disc Reading	Temperature	Barometer	Reading of Disc B	Remarks
	0.55½		A	6.00	5.195								72.0	82.6				82.9	30.6		Fore end towards disc
			31										83.0	72.5							No. 4 Level. Corrn for 1 divn
												Sum	155.0	155.1							for 102.51 chs. [illegible]
												Diff		0.1	0.026	+0.000					" " " " 6 chs. [illegible]
	1.5						C	102.51	1/R	3.330			82.0	73.0						4.310	
													71.2	83.0							
	1.9½						C	"	2/F	3.305			80.0	74.9						4.355	Consider disc as fore
													69.8	84.2							staff. If level correction
November 28th 1900	1.13½						C	"	3/F	3.507			79.5	74.9						4.445	falls in fore end then the
		I											68.0	85.5							correction to fore staff is
	1.17						C	"	4/R	3.183			81.5	72.6						4.280	– and to back staff +
											3.331	3.252	71.0	82.8							and vice versa
												Sum	603.0	630.9							
												Diff		27.9	1.74		−0.080				
	1.19		A/31	6.00	5.195								71.0	82.6							
				Mean	5.195								81.9	72.0							
				+	0.000	5.195						Sum	152.9	154.6							
												Diff		1.7	0.42	+0.001					
	1.22		A/31	6.00	5.195								71.0	82.0							
													82.6	71.0							
												Sum	153.6	153.0							
												Diff	0.6		0.15	−0.000					
	1.23½						C	102.51	5/R	3.216			81.0	72.5						4.370	
													69.0	84.0							
	1.26½						C	"	6/F	2.998			86.0	67.0						4.454	
													77.0	76.0							
do.	1.29	II					C	"	7/F	3.209			80.0	73.2						4.445	
													69.0	83.5							
	1.31½						C	"	8/R	3.154			79.9	73.3						4.422	
											3.144	3.136	68.8	84.0							
												Sum	610.7	613.5							
												Diff		2.8	0.18		−0.008				
	1.33		A/31	6.00	5.185								70.0	82.8							
				Mean	5.190								81.8	71.1							
				+	0.001	5.191						Sum	151.8	153.9							
												Diff		2.1	0.32	+0.002					
	1.38½		A/31	6.00	5.186								72.5	80.1							
													83.5	69.5							
												Sum	156.0	149.6							
												Diff	6.4		1.60	−0.005					
	1.41½						C	102.51	9/R	3.468			78.0	75.0						4.318	
													64.2	88.0							
	1.45						C	"	10/F	3.325			82.3	70.3						4.433	
do.		III											70.0	83.6							
	1.47½						C	"	11/F	3.292			86.0	67.0						4.462	
													72.6	80.5							
	1.49						C	"	12/R	3.126			85.9	67.0						4.357	
											3.302	3.298	72.0	80.9							
												Sum	611.0	612.3							
												Diff		1.3	0.08		−0.004				
	1.51		A/31	6.00	5.201								75.0	78.1							
				Mean	5.194								88.4	64.8							
				−	0.010	5.186						Sum	163.4	142.9							
												Diff	20.5		5.13	−0.015					

Recorded by — Observed by

Computed by — Checked by

results entered in columns 7 and 13, respectively, of the field sheets. The same operations were carried out with the other level from the opposite bank of the river as nearly as possible at the same instants.

The mean of several groups taken in one day constituted the results of a day's observations. Observations were taken on four different days, at the time of minimum refraction, and on two of these days instruments and observers were interchanged.

The following table shows how much the results obtained on different days varied :—

Date.		No. of obs. on each bank		C above B. Feet.
28th November 1900 . . .		30		2·170
30th " " . . .		24		2·052
2nd December " . . .		30		2·202
3rd " " . . .		30		2·105
	TOTAL .	114	Mean .	2·136

It will be noted that, although the mean is very near the figure obtained by the method of vertical angles, the results varied considerably on different days without any apparent reason : there was, even on the best days, a great deal of unsteadiness in the atmosphere, and intersections were always difficult to make with certainty. The disc had to be moved very slowly to prevent overshooting the point of intersection. This was extremely trying to the eyes and, I think, must account for the varying results. If it had been possible to move the foot-screw and so accelerate the intersection by moving the level itself, this might have been obviated; but, as already explained, this would have introduced a very large level correction, which was open to grave objections.

WATER GAUGES.

The measurement of the difference of height by means of water-gauges requires hardly any remark. Two old levelling staves were bolted to piles driven into the bed of the river just below water-level. Simultaneous readings were taken on either side, on two calm days The zeros of the staves were connected with the two bench-marks B and C by spirit-levelling in the ordinary way.

CONCLUSION AND RECOMMENDATIONS.

These experiments seem to me to prove that in the case of a well-selected site, where the current and channel are fairly symmetrical, there exists no great difference of level between the water on the two sides of a river. In this case the difference of level between the bench-marks as obtained by the water-gauge method and the mean of the other two methods, amounts to only 0·076 of a foot or 0·912 of an inch.

As regards the best method of carrying levels of precision across a wide river, I think the method of vertical angles is far preferable to that of levelling. As already explained, it is very difficult to make a satisfactory intersection of a distant object when the object itself has to be moved. The operation is very trying to the eyes, and tedious as well as uncertain. In the case of vertical angles, where the telescope is moved, it is comparatively simple to get a good intersection. Taking into consideration the facts that there is no great difference between the results by the three methods and that there is nearly always a great difficulty in getting a suitable site for observations with water gauges, I would recommend for future work the employment of two 12-inch theodolites with good levels, as being the best and most expeditious method of carrying lines of levels across large rivers.

H. L. CROSTHWAIT.

III

EXPERIMENT TO TEST THE CHANGES IN THE LENGTH OF A LEVELLING STAFF DUE TO MOISTURE AND TEMPERATURE.

BY

J. ECCLES, M.A.

While a discussion was going on as to the best form of levelling staff and the best material for its composition, the idea was put forward that some of the results of levelling in past years might be vitiated owing to the increase in the lengths of the staves due to moisture and possibly also to temperature.

To test the accuracy of the idea, it was decided to make a comparison of a staff during one year. The comparison was commenced on 30th November 1899 and carried on till 7th December 1900, but, unfortunately, the record was broken for two months, April and May 1900, owing to the fact that the miocroscopes used in the comparison had to be removed for use with the Jäderin base-line apparatus. But incomplete as it is, the record is of great interest and sufficient to show the great need for frequent comparison of the staves with the standard bars.

The staff used for the comparison was No. 11 of the ordinary pattern staff used by No. 25 Party for levelling of precision. It is a built up staff consisting of seven strips of teak and two of some soft wood which looks like pine. It was not treated in any way to make it impervious to moisture. Two fine dots were engraved on its brass terminals approximately 10 feet apart.

The staff was placed on two "camels" on the table used for bar comparisons and the standard steel 10-foot bar is placed on two other camels along side of the staff. The staff and bar were brought alternately under the microscopes G and H of the base-line apparatus, and the difference of the distances between the two engraved dots on the staff and the corresponding graduations of the bar were measured in micrometer divisions which were afterwards converted into decimals of a foot.

A wet and dry bulb thermometer and a barometer were placed close to where the comparison was being made, and these were read at the time of the comparison.

The comparisons were made once a week.

The humidity was computed by the usual formulæ, *vis.* :—

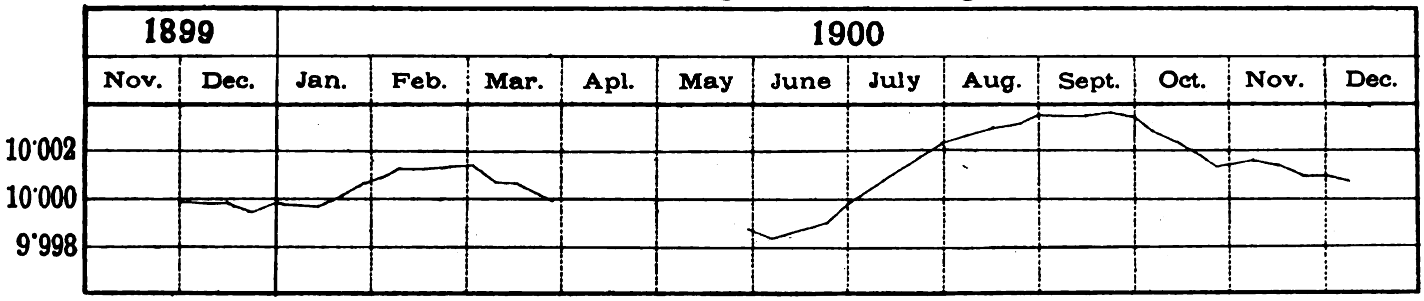
$$f'' = f' - \frac{d}{88} \cdot \frac{h}{30}$$

$$\text{relative humidity} = \frac{f''}{\text{Tension of vapour at temperature of dry bulb}}.$$

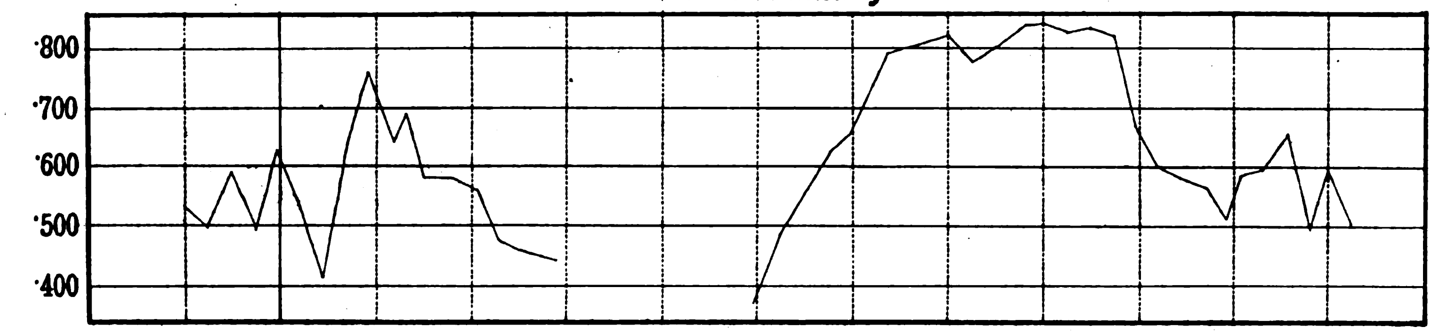
where f'' = tension of vapour at temperature of dew point.

f' = tension of vapour at temperature of evaporation.
d = difference of the readings in degrees Fahrenheit between the dry and wet bulbs.
h = height of barometer.

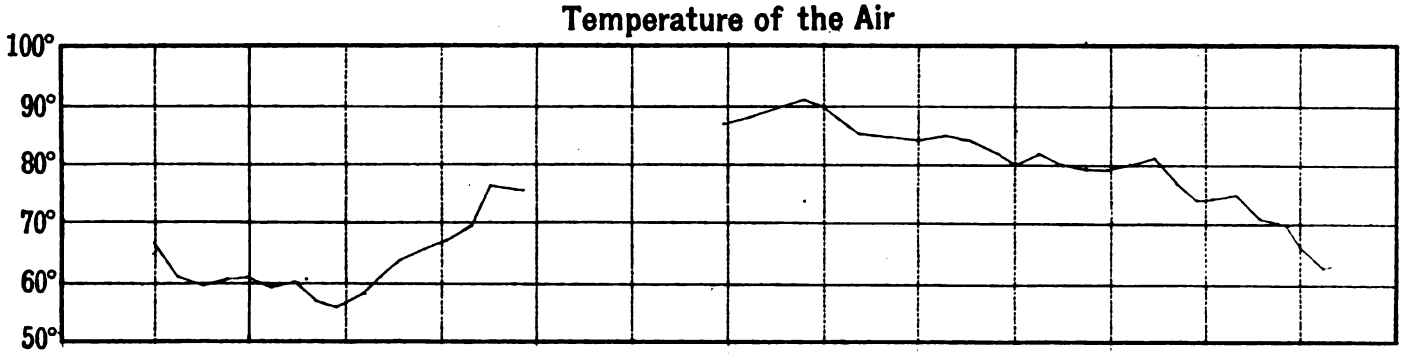
Variation in the length of a Levelling Staff



Relative Humidity



Temperature of the Air



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f′ being taken from the tables given in any of the books on meteorology, such as James′. From the diagram it will be seen that the variation in the length of the staff follows pretty closely on the change of humidity, but does not seem to have much connection with the change of temperature. The staff appears to lag about a week on the humidity. The greatest variation in length is about ·005 of a foot, and it occurs between June and September and corresponds to a change of ·45 in the humidity.

The other great change between February and June is about ·003 of a foot, and corresponds to a change of humidity of about ·30.

Thus, then, during the working part of the year the greatest possible error that could be introduced at one station of observation from the expansion of two such staves would be, when the bottom of one staff was read and the top of another, ·003 of a foot, and, generally speaking, only about half this error. If, however, work is carried on in the rains, an error of ·005 of a foot might be introduced.

It must, of course, be remembered that these errors are to a great extent guarded against by the system of comparing the staves against standard steel bars, but, as in a long line of levels, the total correction for the variation in the lengths of the staves depends on the sums of the rises and the falls, it follows that a very large and appreciable error may be introduced unless careful comparisons are frequently made. For instance, in the line from Bangalore to Mangalore, which is 224 miles long with a difference of level of about 3,000 feet, it was found that the total correction for the expansion of the staves was as much as ·588 of a foot.

It cannot, therefore, be too strongly impressed on levellers that they ought to compare their staves very frequently (at least once a month) against their standard bars, and that, in view of the investigation under discussion, a further comparison should be made about a week after any great change in the weather, either from dry to wet or from wet to dry.

J. ECCLES.

IV

DESCRIPTION OF AN IMPROVED SUN-DIAL FOR USE AT TIDAL OBSERVATORIES.

DESIGNED BY

COLONEL G. STRAHAN, R.E.,

DEPUTY SURVEYOR GENERAL.

When it was first proposed that tide-gauges should be erected at Minicoy and Mergui, it was evident that some difficulty would be experienced in obtaining the local time with sufficient accuracy for the tidal record, as there is no telegraph or astronomical observatory at those places, and the assistants usually appointed to the charge of the gauges are not competent to use the transit instrument or other astronomical methods of finding time with accuracy. The idea occurred to the then Deputy Surveyor General in charge of the G. T. Survey that, under certain conditions, a sun-dial might be designed in which the usual sources of inaccuracy might be eliminated so that its indications would be sufficiently exact for tidal purposes, where extreme precision is not required. A sun-dial, moreover, would have this advantage over astronomical methods that the time may be obtained at once from it by mere inspection without any computation, except the very simple one of the "equation of time" which is necessary for reducing apparent solar time to mean solar time, this equation being a quantity that may easily be tabulated for daily, or even for hourly, use at any place whose longitude is known.

The most casual observer with a sun-dial of the usual pattern will soon appreciate the difficulty that arises in knowing exactly where the edge of the shadow cast on the dial plate really begins. If the sun were like an electric arc light, in which the actual point of brilliance is extremely small, there would be no difficulty, as the edge of a shadow cast by such a light is hard and well-defined to an extraordinary degree, but as the apparent diameter of the sun is about 30′, and that body takes, in round numbers, about two minutes of time to pass over a space in the heavens equal to its own diameter, the obvious result is a penumbra on the dial plate involving an uncertainty of nearly that amount in the estimated time. This, and the difficulty of placing a sun-dial correctly in its proper position, constitute the only drawbacks to its use. The first of these difficulties, which is the more serious one, is met, in this improved sun-dial, by the use of a narrow slit, through which the sun shines, and casts a thin line of light on the dial plate, in place of the edge of a shadow cast by a solid object. It seemed almost certain that the centre of a line of light thus formed could, in spite of its penumbral edges, be estimated with very considerable precision, and experience showed that this really is the case, and that independent observers, when asked to point out where, in their judgment, the centre lay, differed almost inappreciably in their estimates of its position. The second difficulty was met by a simple little piece of mechanism which will be explained hereafter.

The design of the instrument, as described below, is better adapted for low than for high latitudes. Under the latter condition the dial would be tipped up at such a high angle that its centre of gravity would come perilously near the edge of the base, and some special arrangement or counterpoise would be necessary to make it secure.

The essential parts of this sun-dial, of which a drawing not made to scale is given in Fig. 1, should be made of metal (brass will suffice for all but the scale) and are eight in number, *viz.*:—

1. The upper plate A carrying the slit a.,a.
2. The base plate or dial plate B.
3. The four pillars or supports C, C, C, C.
4. The foot-screws D, D, D.
5. The pedestals E, E, E.
6. The detached scale F.
7. The levelling wedge and rectangle G.
8. A small level not shown in the diagram.

A little explanation with regard to each of these parts is necessary, and it should be borne in mind that the dimensions given need not necessarily be adhered to. They are merely noted here as being very approximately those employed in the instruments actually in use with the Tidal Party. The upper plate A should be 4 inches long, 1½ inches wide, and $\frac{1}{10}$ of an inch thick, and the slit, which should be centrally cut in it, should be 3 inches long and $\frac{1}{40}$ inch broad. The edges of the plate where the slit is formed must be bevelled off, otherwise the sun would only be able to shine through the slit when it was on, or very near to, the meridian. A transverse section through the upper plate would therefore be as in Fig. 2. The base or dial plate B, which should be a perfect plane, is 8 inches square and $\frac{1}{8}$ inch thick, and has three projections on it, either firmly attached to it, or preferably cast with it. Of these projections or ears, one is fixed at the centre of the north end of the plate, and the other two are attached to the corners of the south edge. These projections are designed to hold substantial foot-screws which come into use when the dial is to be adjusted. For appearance sake it is better that these projections should not be exactly in the plane of the plate, but just so much bent that they may remain horizontal, or approximately so, when the dial is raised to the latitude for which it is designed.

The four pillars or supports are for binding together in an invariable position the upper and base plates, and stiffness in this respect is of great importance. Their section should be a square of about ¼ inch (or more) to the edge, and the way in which the plates should be attached is shown in Fig. 3, which is not drawn to scale. The length of these pillars is to be such as will give a clear distance of 8·6 inches between the lower surface of A and the upper surface of B.

The plate A must be centrally placed over B, but, except for the sake of symmetry and appearance, the exact parallelism of the plates is unimportant, so long as care is taken that the slit is parallel to the dial plate.

With regard to the foot-screws it is only necessary to remark that they should be substantial enough to be free from shakiness, ½-inch diameter would probably be sufficient, the length should be 1 inch, with milled heads 1 inch in diameter.

The pedestals are used merely for the foot-screws to stand upon. They may be 2½ inches diameter, ¼ inch thick, with a transverse groove cut in them for the foot-screws to rest in, and they should be placed as shown in Fig. 4. The scale F, *vide* Fig. 5, now claims our attention. This may be made either of metal or ebonite, wood would be too sensitive to hygrometric changes, and would be easily liable to damage. The expansion and contraction of metal or ebonite by changes of temperature is far too small to be of any consequence. If the scale is of metal it should be painted black with white divisions, or *vice versâ*, and it is important that it should present a dead surface, for a polished or reflecting surface would materially interfere with precision in noting the position of the image of the slit. For the sake of brevity this bright line cast on the dial plate by the sun shining through the slit will be called the "index" throughout this paper. This scale is rectangular in form, 7½ inches in length, 1 inch in breadth, and $\frac{1}{10}$ inch thick. On the upper face both the edges should be bevelled for about ⅛ inch at the centre of the scale to enable its zero to be accurately adjusted to the meridian line on the dial plate, of which meridian line more will be said when the adjustments of the instrument are considered. The scale to be engraved on it is not a scale of equal parts, but a scale of natural tangents to radius 8·5 inches. The length of this radius, *viz.*, 8·5 inches, is important, being exactly the distance between the slit and the *upper* surface of the scale. Whatever dimensions may be selected for the dial, it is indispensable that the radius should be equal to this distance. The divisions correspond to solar hour angles of 30′, 1°, 1° 30′, 2°, and so on, reckoning from the centre both ways. The divisions at 2° 30′, 5°, 7° 30′, and further multiples of 2° 30′, should be engraved completely across the scale, the intermediate ones should be about half this length, and lie centrally along the scale, as shown in the diagram.

Each long graduation will then signify some multiple of 10 minutes of time, and should be numbered accordingly, the intermediate graduations representing 2 minutes each. Only the central part of the scale is shown in the figure, it will in reality extend to about 1¾ hours on either side, and it follows, therefore, that the observation for time must be made when the sun is within this distance of the meridian, either before or after noon. The divisions should be in fairly thick lines, and the observer should wait until the index is centrally over one of them; this waiting cannot, at most, entail more than a delay of two minutes. The method of laying out this scale may safely be left to the instrument makers, but it would probably be the best plan to compute from a table of natural tangents the exact distance of each division (or perhaps each alternate division) from the centre in inches and decimals, and thus mark them off. As a guide to the refinement required for the graduation, it can be easily shown that towards the middle part of the scale an error of $\frac{1}{100}$ inch in round numbers in the position of the stroke would entail an error of ten seconds in time, and as this is about the limit of precision to be expected from the instrument, greater accuracy than this seems unnecessary.

Before describing the wedge and rectangle it will be as well to explain the adjustments required in this sun-dial before it can be used for obtaining time. These adjustments are four in number, and are as follows. Firstly, a zero or meridian line must be engraved on the dial plate, fulfilling the two conditions that it must be in the same plane as the longitudinal axis of the slit, and also that this plane in which the zero line and the axis of the slit lie must be perpendicular to the plane of the dial plate: its perpendicularity to the plane of the upper plate is immaterial. This adjustment must be made once for all by the maker, and it is impossible that it can, when once made, be thrown out except by very rough usage, as the frame-work is all designed stiff enough to prevent change. Secondly, the zero line must be raised at its northern end until its inclination to the horizon is equal to the latitude of the place. This is done approximately in the construction of the masonry pillar on which the dial stands, as seen in the diagram, the final adjustment being completed by the foot-screws with the help of the wedge and rectangle, as explained below. Thirdly, the plane in which the slit and the zero line lie must be made truly vertical. Fourthly, the zero line must be truly adjusted in azimuth.

To perform the second and third adjustments a piece of mechanism, G, consisting of a metal rectangle 4 inches long, 1¼ deep, and $\frac{4}{10}$ of an inch thick, united at its centre to a wedge of the same thickness but with length adapted to the particular latitude for which the dial is designed, is employed. It must be very accurately squared and planed, and the angle of the wedge must be equal to the required latitude. This little apparatus, marked G in Fig. 1, is supported on the slope of the dial plate by two little studs so placed that the line joining them is truly perpendicular to the zero line. A small level accompanies the mechanism, and it is obvious that the second adjustment before-mentioned is secured if the upper surface of the wedge is levelled by means of it, using the two foot-screws at the north end of the plate, the precaution of reversing the level end for end being of course observed, and also that the third adjustment depends similarly upon levelling the top of the rectangle by means of the two foot-screws at the south end. There remains, then, only the fourth adjustment, *viz.*, bringing the zero line into its true position, and the easiest means of doing this is to set it by means of a clock or chronometer which can be relied on. All that is necessary, is, that at the instant of apparent noon the whole instrument should be gently moved round until the index lies along the zero line on the plate. This may slightly derange the second and third adjustments, so that it may be necessary to repeat the process.

It seems almost superfluous to give any details as to the construction of the pillar; the pattern shown in the diagram proved satisfactory in actual use, and is perfectly simple and easy to construct.

If a sun-dial of this design is intended to be used at any other station whose latitude differs from that for which it was originally made, the only change necessary is in the wedge and in the height of the step on the pillar.

When using the instrument, the observer must lay the scale carefully down on the dial plate with its zero exactly coinciding with the zero line, and in contact with any one of the pairs of studs let into the plate which is found convenient. These studs are placed so that the line joining each pair may be truly perpendicular to the zero line. In the winter, owing to the southern declination of the sun, the index will fall towards the north end of the plate, and in the summer, in the opposite direction, and a suitable pair of studs must be selected accordingly. The observer will then watch the index until he sees it exactly bisected by one of the scale divisions, and the reading of the scale at this instant determines the apparent sun's hour angle, either before or after its meridian passage. In the former case this hour angle must be deducted from 24 hours, and the remainder is the apparent time; in the latter case the hour angle is itself the apparent time, from which, by the proper application of the equation of time, the mean solar time is at once deduced.

It is desirable that, when the position of the dial is once satisfactorily adjusted, the pedestals should be fixed with some mortar or cement, so that if it is found necessary to remove it for any purpose, it may be replaced in its former position without the trouble of re-adjustment.

When not in use, a wooden cover with lock and key should be placed over the instrument, as its adjustments might easily be disturbed by malicious hands without the observer becoming aware of it.

The foregoing description, with very trifling exceptions, applies to instruments which have been actually in use with the Tidal Party, and have been found to work efficiently, but modifications of this original design suggest themselves which it might be advisable to introduce in future sun-dials made for a similar purpose.

The essence of the instrument is the slit, adjusted in as nearly as possible perfect parallelism with the earth's axis. The manner of receiving this on a scale to show the sun's hour angle may be varied in many ways.

If the slit is made sufficiently long, a moveable scale is no longer necessary, for it can be so contrived that whatever be the sun's declination either north or south, some part at least of the index will fall on a scale fixed permanently at the middle of the dial plate, thus obviating the necessity of a moveable scale and the studs on which it rests.

To show this, let AB in Fig. 6 be the upper plate (shown in side elevation), SN the lower one, and DE the transverse section of the scale; then when the sun is at its furthest southern declination A B D F will represent the illuminated plane of solar rays shining through the slit, and by a proper adjustment of dimensions the scale DE can be included in the illumination, and similarly for the sun when at its greatest northern declination. If we take this declination to be 23°, the distance between the upper surface of the base plate and the lower surface of the slit to be 8·5 inches, and the breadth of the scale 1 inch, then it is easily seen that $AB = 1 + 2 \times 8{\cdot}5 \times \tan. 23^\circ = 8{\cdot}2$ inches, and therefore, that with a slit of this length and a distance of 8·5 inches between the plates, the "index" will at all times of year fall on the scale, but it would still be necessary for the purposes of adjustment, as before explained, that the zero line should be engraved on the lower plate.

A further modification might be made if the division of the scale into natural tangents, instead of into equal parts, should be considered objectionable. This could be done by making it in the shape of a circular arc, or perhaps it would be more exact to describe it as a portion of a cylinder with the slit as its axis. The adjustment might be perhaps a little more difficult in this case, but there would be the advantage that the "index" would not become spread out or ill-defined as the sun receded from the meridian. Experience, however, shows that this objection is of no account so long as the sun is within two hours of the meridian on either side. If it were considered desirable, on account of the appearance of the dial, to make the lower plate horizontal, a slight computation would enable the maker to engrave a suitable scale upon it, but the advantage would be doubtful as the instrument would then be only suitable for the precise latitude for which it was made, and, on the whole, it seems that the original form of the dial is as well suited to its purpose as any of the modified arrangements.

G. STRAHAN.

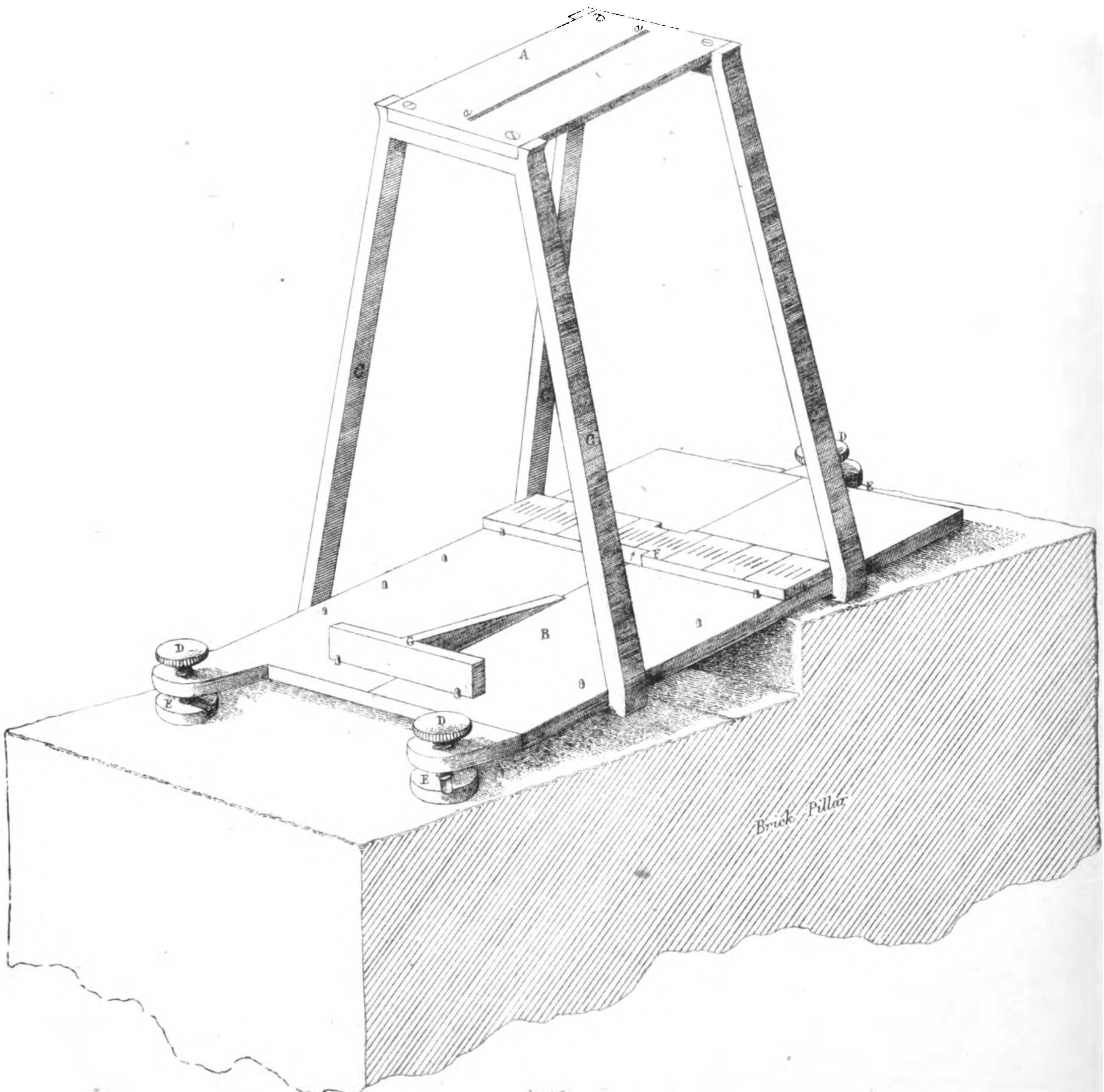

Fig 1.

Fig. 2

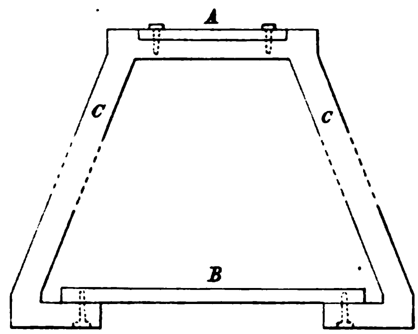

Fig. 3

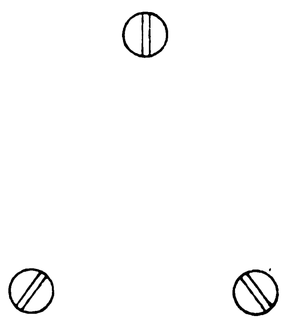

Fig. 4

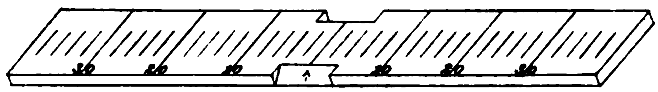

Fig. 5

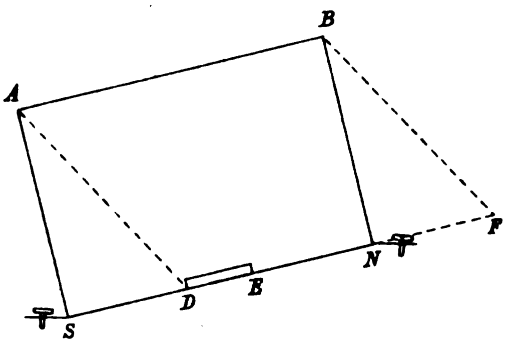

Fig. 6

V

NICKEL-STEEL ALLOYS AND THEIR APPLICATION TO GEODESY

BY

M. CH. ÉD. GUILLAUME.

1ST PART.

PROPERTIES OF NICKEL-STEEL.

Preliminary.—The late Dr. J. Hopkinson was the first to draw attention to the singular magnetic properties and *permanent* changes of volume of alloys of steel and nickel containing about 25 % of the latter.

In 1895 M. Benoît discovered that an alloy containing 22 % of nickel and 3 % of chrome was as extensible as brass. The following year I established the fact that an alloy with 30 % of nickel was less extensible than platinum. These different results from measures taken at random led me to pursue the study of the curious anomaly they presented. The researches, undertaken with the co-operation of the Society of Commentry-Fourchambault, extended to variations of volume, magnetic properties, elasticity and electric resistance of alloys of steel and nickel, as well as to their resistance to oxydation and the ease with which they could be worked mechanically,—points particularly important from a geodetic point of view.

The result of the researches shows that the anomalies of nickel-steel alloys are so related among themselves that, knowing them for one alloy, it is possible to a certain extent to predict them for others,—as I have pointed out in different publications from time to time. I wish to limit myself here to a description of the properties, useful directly in the construction of instruments.

It is sufficient to say here that in all their properties these alloys can be placed in two great classes called *reversible* and *irreversible*. To the first belong the alloys containing more than 25 % of nickel, characterized by the fact that their properties, at least to the first approximation, only depend on their actual temperature, while the properties of the second class of alloys can differ entirely according to what the preceding cycle of temperatures was. Thus, at the ordinary temperature, an alloy containing 24 to 25 % of nickel is non-magnetic, soft and as extensible as brass when it is brought from a high temperature, while if it has passed from a low temperature, it is strongly magnetic, very elastic, and about as extensible as steel.

NOTE.—This paper appeared in the *Report of the 13th Conference of the International Geodetic Association, 1901.*

The change from one state to the other which takes place gradually as the temperature is lowered, is accompanied by an increase of volume about 2 per cent. These last alloys have not so far been employed in Geodesy, and I will make no further remarks about them.

Reversible Alloys.

Magnetic Properties.—All the alloys in this class are magnetic at low temperatures and non-magnetic at high ones. The change from one state to another takes place not far from 0° C. for an alloy containing 26 to 27 %, of nickel, and the temperature of the change rises with the increase of nickel at first 30° to 35° for each 1 % of nickel, then slower and slower. The loss of magnetism is gradual, and its diminution can be observed at more than 200° C. from the vanishing point.

Expansions.—In the non-magnetic state the expansions of all these alloys are much higher than those of their constituents: they are, in fact, in most cases almost equal to or even greater than the expansion of brass. But after the beginning of the magnetic region has been reached by cooling, the contraction which is under determination diminishes progressively, and for some alloys soon reaches a very small value. The alloy is then in a state of deformation which is thermically reversible, and passes by heating through the same phases as it did in cooling.

Starting from 25%, if we increase the amount of nickel we find the alloys are at first very expansible, then the expansion decreases gradually and rapidly, passes towards 36% through a minimum, and then increases more slowly. The expansions do not depend entirely on the amount of nickel but also on additions of carbon, manganese, chrome, and, finally, on the mechanical work done on the alloy when hot or cold.

The table given below, constructed from measures taken in 1896 on the first alloys made, can only, therefore, be considered as an example. In the numerous castings made since that time it has not always been possible to get the lowest expansions indicated here, while, on the contrary, with particular precautions, expansions $\frac{1}{8}$ or $\frac{1}{7}$ of these have in exceptional cases been obtained. On the other hand, the co-efficients of the second term in the formula have always come out the same within the limits of the errors of observation:—

Percentage of Nickel.	Co-efficient of mean expansion between 0 and θ°C.
26·2	$(13·103 + 0·02123\,\theta)\ 10^{-6}$
27·9	$(11·288 + 0·02889\,\theta)$,,
28·7	$(10·387 + 0·03004\,\theta)$,,
30·4	$(4·570 + 0·01194\,\theta)$,,
31·4	$(3·395 + 0·00885\,\theta)$,,
34·6	$(1·373 + 0·00237\,\theta)$,,
35·6	$(0·877 + 0·00127\,\theta)$,,
37·3	$(3·457 - 0·00647\,\theta)$,,
39·4	$(5·357 - 0·00448\,\theta)$,,
43·0	$(7·452 - 0·00312\,\theta)$,,
44·4	$(8·508 - 0·00251\,\theta)$,,
48·3	$(9·843 + 0·00013\,\theta)$,,
* 100	$(12·514 + 0·00674\,\theta)$,,
34·8 + 1·5 Cr.	$(3·580 + 0·00132\,\theta)$,,
35·7 + 1·7 ,,	$(3·373 + 0·00165\,\theta)$,,
36·4 + 0·9 ,,	$(4·433 - 0·00392\,\theta)$,,

If we compare the numbers in this table with the expansions of the metals and usual alloys, we see that the series of new alloys gives at first expansions which are for the most part about the

* Pure nickel of commerce—means of several samples.

same as those of the metals and usual alloys, then at 29 % the expansion passes below the smallest known expansion to reach at 36 %, one-tenth the expansion of platinum and one-twentieth that of brass. This last alloy has been called *Invar* by M. Thury. Fig. 1 shows all the results obtained in the measures with the "comparator" or in other measures by a particular method in which a temperature of over 200° C. can be obtained.

Mechanical Constants.—The reversible nickel-steels are relatively soft at the annealing stage, but they can be brought by hammering to an elasticity which renders them free from permanent deformation after ordinary strains. Some of them, especially those with chrome added, can be made into springs quite good enough for many purposes.

The density varies in the same way as the elasticity, that is to say, in the case of alloys of small expansion there is a slight departure from the law of mixtures. Between 30 % and 40 % the density is always between 8·0 and 8·1.

The modulus of elasticity follows, though in a lesser degree, the course of the expansions. Between 26 % and 36 % it passes from 18·5 to 15·0 tons per square millimetre and rises a little after the last percentage. The deformations of the less expansible alloys are, therefore, for equal tensions about a ⅓rd part greater than those of hard steel or nickel, and a little greater than those of soft iron: chrome raises the modulus of elasticity.

Transitory and Permanent Variations of Volume.—The reversible nickel-steels show variations with time or under the influence of changes of temperature which it is essential to take into account both in the construction or preparation of standards for practical use and in the measures for their preservation or the conditions of their employment in the field. It is necessary, therefore, to study these variations with care.

An alloy of this kind brought quickly from a temperature θ_1 to another θ_2 does not immediately take up the length corresponding to the actual temperature but reaches it gradually often after a rather long time.

When it passes to a temperature higher than that at which it had taken its state of equilibrium, the alloy undergoes, as the temperature settles, an expansion corresponding to the percentage of nickel—then it contracts slowly afterwards.

Inversely, after being contracted, it increases slightly in length.

It is important to know first the difference between the states that are observed at the very moment of the establishment of the new temperature and those which are established after a very long time, and then the rapidity with which the passage from one state to another takes place at each temperature.

I have determined the elements of the transitory variation for a great number of specimens of the less expansible alloys. The succession of states obtained, whether by a rapid change or by a slow variation of the temperature is shown by the curve of Fig. 2. The abscissæ are the temperatures, and the ordinates the residual variations, in microns per metre. We see, for example, that if a bar which has attained its perfect equilibrium at 0° C.. is suddenly raised to 40° C. it ought to contract about 5 millioniemes. As the curve is one of great curvature, the changes corresponding to the ordinary changes of the surrounding temperature are much smaller. Between 0° C. and 100° C. these variations are sufficiently well represented by

$$\Delta l = -0{\cdot}00325\ 10^{-6}\ \theta^2.$$

θ being the temperature reckoned from zero.

On the subject of the rapidity of the variation I will only quote the following figures:—

When the temperature passes from 10° C. to 25° C. we begin to perceive a difference at the end of 20 hours and the movement which has a total of about 1·5 millioniemes is entirely finished in

200 hours. If we pass from the ordinary temperature to 100° C. we find a contraction of 10 millioniemes at the end of 10 minutes and the total movement is completed in less than an hour. When the temperature is lowered the movements are at the same final temperature much slower than in the first case.

If, for example, we pass from 100° C. to 60° C., we find the movement is in the beginning about $0\mu\cdot1$ per hour for a bar 1 metre long, and only stops in about 300 hours. From 60° C. to 40° C. the initial movement is $0\mu\cdot025$ per hour, and only stops after 700 hours.

At the ordinary temperature the movement is much slower. Besides the variations which take place slowly and finish after a considerable length of time we find others which appear to differ in principle from the first, and consist in a slow and gradual lengthening of the bars which may go on for years, and which gradually tends to a limit. These movements take place more rapidly when the temperature is high, and only complete their cycle when the alloy has been subjected to a series of annealings at gradually descending temperatures.

Fig. 3 shows the variations undergone by a bar of the less extensible nickel-steel annealed slowly from 150° C. to 40° C., then left for several years at the temperature of the laboratory. The points of the diagram represent the lengths observed directly at the temperature of 15° C. to which the bar was always brought for measurement. These points form a series of annual festoons falling as the surrounding temperature (whose variations are indicated by the dotted curve above) rises. The continuous curve which shows the variations in regard to time is made by the points corresponding to the lowest surrounding temperatures. The distances between the isolated points and the curve correspond sensibly to depressions due to variations of temperature such as are represented in Fig. 2.

The continuous curve shows that at the end of 3 years of rest the bar had increased a little less than $0^{mm}\cdot01$. In the third year the variation was not more than about 1μ.

These variations can be still reduced if, before laying the bar aside in the temperature of the surrounding air, it is kept for some weeks between 25° C. and 30° C.

The variations mentioned refer to an alloy of about 36°/₀ of nickel: as the percentage is increased, these variations diminish. At 30°/₀ of nickel they are much greater than the above-mentioned, and at 45°/₀ they are inappreciable.

Different Properties.—The inexpansible alloys from good castings are almost exempt from bubbles: they take a good polish, and are susceptible of receiving outlines of great fineness and perfection.

The alloys are unchangeable in damp air, and well polished surfaces are only slowly affected in cold water; on the other hand, they are attacked by acids which corrode them very much.

The reversible nickel-steel can be rolled and wire-drawn, and can be worked easily enough with a plane or lathe if we take small cuts with strong well-tempered tools, working slowly. They are easily attacked by a file, but they spoil it quickly. Working up the surface of pieces cast or rolled hot blunts the hardest tools after a time.

2ND PART.

APPLICATIONS OF NICKEL-STEELS TO GEODESY.

We have just seen that certain nickel-steels containing about 36% of nickel possess an expansibility about $\frac{1}{10}$ of that of platinum, rendering practically negligible the errors due to an insufficient knowledge of the temperature of the standards, in the ordinary conditions in which they are employed. None of their other properties forbid their use in circumstances where their small expansion would suggest their employment beforehand, and even certain of them are more advantageous than the corresponding properties of most of the metals and usual alloys.

For example, the small amount of oxidation to which these alloys are liable, removes one obstacle to their employment which exists in the case of steels in general, and the price of nickel-steels, though considerably higher than that of steel, will never be considered among the expenses of a geodetic campaign. Volume for volume the price of the alloys is about $\frac{1}{300}$ of that of platinum, so that the great objection to the employment of platinum in large sections does not exist in the case of invar. In practice the above ratio is too small owing to the value of the waste in the case of platinum, but even if we treble it, the final cost of invar is a trifle compared with platinum.

Their want of stability may be urged against the nickel-steels, and it may be feared that their slow variations may cause regretable uncertainty as to the lengths of the geodetic standards. This will certainly be the case if the alloys have not been subjected to the series of annealings necessary to bring them to their greatest stability, or if, when employing them, account is not taken of the law of these variations.

We may, however, remark that when the percentage of nickel distinctly passes 40% the stability increases rapidly, and that towards 43% to 44% it rivals that of the pure metals and of the most stable known alloys with the exception of platinum-iridium. Anyhow, we can assert that a bar of one of these alloys does not vary more than 1 millionieme in the two or three years after its manufacture. For Geodesy, then, these alloys may be considered absolutely invariable, and their expansion, a little inferior to that of platinum, renders them preferable to the latter when we take the cost into consideration.

Alloys, then, of 43% or 44% of nickel completely take the place of platinum as far as geodetic measurement is concerned, and make it possible, thanks to their price, to construct extremely rigid standards of measurement.

But if we look closer into the matter we see that the variability of the less extensible alloys is not a serious obstacle to their employment in Geodesy.

I presume that every geodetic service possesses the means of comparing its standards before and after the campaign which does not begin till at least a year after the standard is made, and does not last more than one or two years.

Let us return to Fig. 3, which represents the variation in the course of years of a bar 1 metre long left at the temperature of the surrounding air, and let us consider the variations which take place after the first year. In the two years which follow, the variation is very little more than 2μ so that if we take a mean for the whole campaign, we shall not commit an error sensibly greater than 1 millionieme, and if we interpolate with regard to the time, the error will be still less. On this slow variation it is true there is superimposed another which depends on the variations of the surrounding temperature. But the standard, if it is preserved under proper conditions, will not follow these variations to their full extent and the temperatures read by the accompanying thermometers may be considered in general as slowly reached If, therefore, we incorporate in the formula for expansion the corrective term due to the prolonged exposure to temperatures, we shall have taken sufficient account of the annual variations. In the vicinity of 20° C. it would take an error of about 8° C. in the mean temperature to falsify by 1 millionieme the length given by the formula for expansion thus corrected.

Nevertheless in a very long campaign or in one where the temperatures reached considerable values, say for example over 30° C., it would be necessary to carry another standard, such as one of Brunner's bimetallic bars, and compare it and the nickel-steel standard employed in the measurement under the best possible conditions as regards temperature, in a large underground room for example. In such conditions the faults which bimetallic bars present when employed in the field will be considerably lessened, and they will afford a very convenient standard of reference if, in spite of their delicacy of construction the conditions of transport allow us to guarantee their perfect preservation. Nevertheless, even for such a purpose, it would seem preferable to use bars of very stable nickel-steel, large in section and properly protected.

The Bimetallic System.—In a monometallic system the errors due to an imperfect knowledge of the temperature are directly proportional to the expansion of the bars. In a bimetallic system the relations are more complicated, and require a closer study. Consider two standards whose coefficients of expansion are, respectively α_1 and α_2; let l_0 be their common length at a temperature which we will assume is 0° C., l'_θ and l''_θ their respective lengths at θ° C.

This temperature, which we can deduce from the measures made, by the same means, will be given by

$$\theta = \frac{l''_\theta - l'_\theta}{l_0 (\alpha_2 - \alpha_1)} \text{ when } \alpha_2 > \alpha_1,$$

from which the length of the less expansible bar, considered as the principal standard, will be

$$l'_\theta = l_0 \left\{ 1 + \alpha_1 \frac{l''_\theta - l'_\theta}{l_0 (\alpha_2 - \alpha_1)} \right\}.$$

If, in reading the distance to be measured, we have made errors $\Delta l'$ and $\Delta l''$ respectively with the two standards, there will result in the determination of l' the errors $\Delta l' \frac{\alpha'}{\alpha_2 - \alpha_1}$ and $\Delta l'' \frac{\alpha'}{\alpha_2 - \alpha_1}$. The first ought to be added to the error already committed which, if it is positive, will give too small a value to the section measured on the base. The error of reading of l' will be therefore finally multiplied * by $\frac{\alpha_2}{\alpha_2 - \alpha_1}$ in the result.

The error made on each of the standards is either due to observation or temperature: the first is independent of the nature of the standard, and the second is proportional to the expansion. A numerical example will better show the advantage in a bimetallic system of employing an inexpansible principal standard.

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* *Translator's note.*

Let Θ be the temperature determined by the lengths $l'_\theta + \Delta l'$ and $l''_\theta + \Delta l''$:

$$\left.\begin{array}{l} \text{then } l'_\theta + \Delta l' = l_0 \{ 1 + \alpha_1 \Theta \} \\ \text{and } l''_\theta + \Delta l'' = l_0 \{ 1 + \alpha_2 \Theta \} \end{array}\right\} \quad \therefore \quad \Theta = \theta + \frac{\Delta l'' - \Delta l'}{l_0(\alpha_2 - \alpha_1)}.$$

$$\therefore \text{ New length of the } l_1 \text{ bar} = l_0 (1 + \alpha_1 \Theta)$$

$$= l_0 (1 + \alpha_1 \theta) + \frac{\alpha_1}{\alpha_2 - \alpha_1} (\Delta l'' - \Delta l')$$

$$\therefore \text{ Error due to wrong temperature resulting from wrong length } = \frac{\alpha_1}{\alpha_2 - \alpha_1} (\Delta l'' - \Delta l').$$

$$\therefore \text{ Correction to } \Delta l' \text{ due to wrong temperature } = - \frac{\alpha_1}{\alpha_2 - \alpha_1} (\Delta l'' - \Delta l').$$

$$\therefore \text{ Error in } l' = \Delta l' - \frac{\alpha_1}{\alpha_2 - \alpha_1} (\Delta l'' - \Delta l')$$

$$= \frac{\alpha_2}{\alpha_2 - \alpha_1} \Delta l' - \frac{\alpha_1}{\alpha_2 - \alpha_1} \Delta l''.$$

The table below contains this indication on the suppositions that the principal standard is invar of these different qualities, platinum-iridium, iron or nickel, the other standard being nickel-steel of mean expansibility or brass. For each case the table contains in its four columns the factor which multiplies the error of reading $\Delta l'$ and $\Delta l''$ of each of the standards in the final result, the error in millioniemes resulting from an error of 1° C. in the temperature of * the standards ($\Delta\ \theta$), finally, this error referred to the case of actual bimetallic bars, platinum iridium-brass, to which we attribute the coëfficient 1 ($\delta\theta$). The expansions are given in millioniemes per degree.

	α_1	Nickel-Steel, $\alpha_2 = 9$.				Brass, $\alpha_2 = 18$.			
		$\Delta l'$	$\Delta l''$	$\Delta\theta$	$\delta\theta$	$\Delta l'$	$\Delta l''$	$\Delta\theta$	$\delta\theta$
Invar	0·5	1·06	0·06	0·53	0·03	1·03	0·03	0·51	0·03
	1·0	1·12	0·12	1·12	0·07	1·06	0·06	1·06	[illegible]
	1·5	1·20	0·20	1·80	0·11	1·09	0·09	1·64	0·10
Pt. Ir.	8·6					1·91	0·91	16·41	1·00
Iron	11·5					2·77	1·77	31·86	1·94
Nickel	12·5					3·27	3·27	40·85	2·49

There would be no use in employing any of the last three metals with an alloy with an expansion of 9 millioniemes. A comparison of the other numbers shows that the errors of reading made on the invar standard are multiplied by a coefficient which in the most unfavourable case is only 0·2, and when the secondary standard is brass, never reaches 0·1. When platinum-iridium and brass are used the factor is 1·91, and increases very rapidly for the more expansible metals.

The error of reading of the more expansible standard becomes negligible when it is associated with an invar standard, it is very little diminished when it is employed with a platinum-iridium one, and greatly increased when associated with an iron or a nickel one.

The errors of the result depending on the temperature show better still the advantages of an inexpansible standard. Thus, taking as unity the only combination hitherto used in the measurement of bases, we see that the errors resulting from the employment of invar would be reduced to quantities varying between 0·03 and 0·10, while they are hardly less than 2 when iron and brass are used.

We see also that if the principal standard is made of a very inexpansible alloy, there is no advantage in constructing the other of a very expansible metal, as an alloy of mean expansibility gives practically the same results. We should therefore be guided in the choice of the latter by practical considerations, permanence with time, facility of working, resistance to inclemency of the weather, and, for wires, elasticity and facility in rolling, etc.

* *Translator's note.*

The error in $l'\ l''$ due to an error of 1° in the temperature of $l' = \alpha_1\ l_0$,

and the multiplier is $\dfrac{\alpha_2}{\alpha_2 - \alpha_1}$ (see note on page 24):

$$\therefore\quad \Delta\theta = \alpha_1 \frac{\alpha_2}{\alpha_2 - \alpha_1},\ \text{if } l_0 \text{ is 1 metre.}$$

If there is also an error of 1° in the temperature of l'', $\Delta\theta$ will vanish, so that really $\Delta\theta$ represents the effect of 1° error in the difference of temperatures of l' and l''.

$$\delta\theta = \frac{\Delta\theta}{16{\cdot}41},\quad 16{\cdot}41 \text{ being the value of } \Delta\theta \text{ for Pt.-Ir.}$$

In M. Jäderin's system the principal error is due to the want of knowledge of the temperature: the employment of invar of the best quality ensures an elimination of errors of temperature about 50 times better than the usual combination.

CONCLUSIONS.

We see from what precedes, that considerable advantages result in the measurement of bases from the use of the new alloys. These advantages are only limited by the slow and residual variations to which it is necessary to pay the greatest attention but which can be rendered almost inoffensive if we follow carefully the indications of experience. Certain applications of the alloys have already been made. The Geographical Service of the French Army possesses wires of the least expansible alloy, and have been able to test them. The Swedish-Russian expedition to Spitzbergen has taken them for their measurements and the results obtained by M. Jäderin are very favourable to the employment of the new wires. Dr. Gill has adopted them for the geodesy of Cape Colony, and the results of these operations, so far as is known, do not give any cause for disappointment regarding the permanence of these wires. Finally, the U. S. Coast Survey have employed these alloys for their theodolites, and express themselves quite satisfied.

For circles, for example, when slow homogeneous deformations have not any effect on the measures, the very inexpansible alloys present nothing but advantages. They allow the division to be traced directly on the limb much easier than can be done in the present system of graduating on silver. We may therefore assert that when the obstacles which for a time always stand in the way of any thing new have been removed by the force of circumstances, the nickel-steels will be in general use in Geodesy.

Translated by
J. ECCLES.

Fig. 1

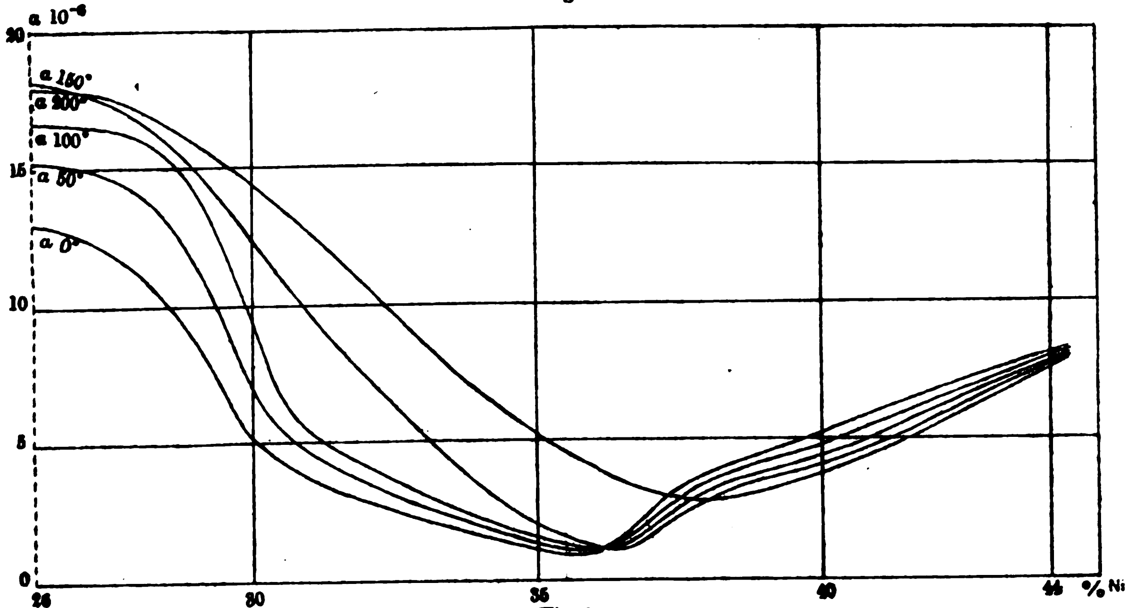


Fig. 2

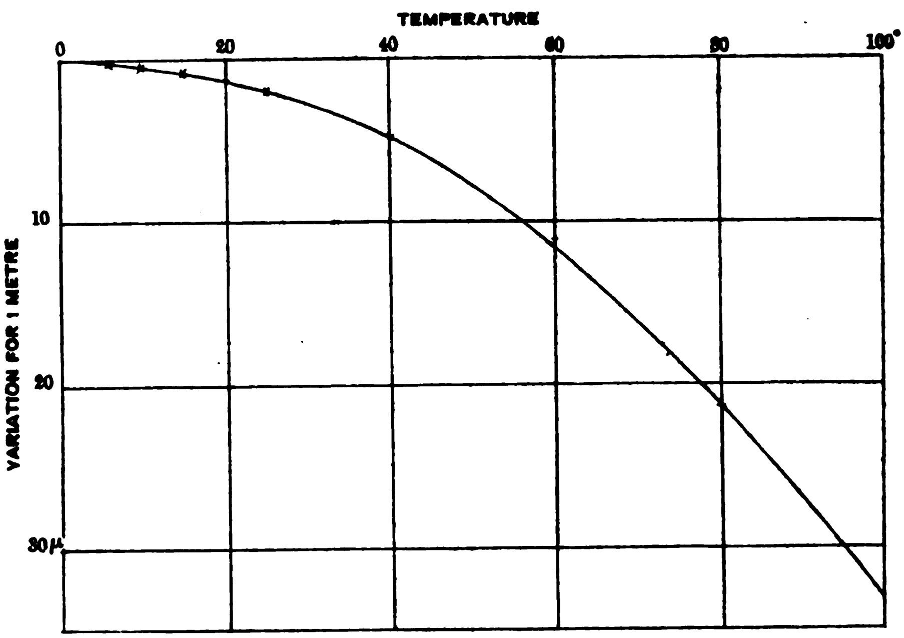
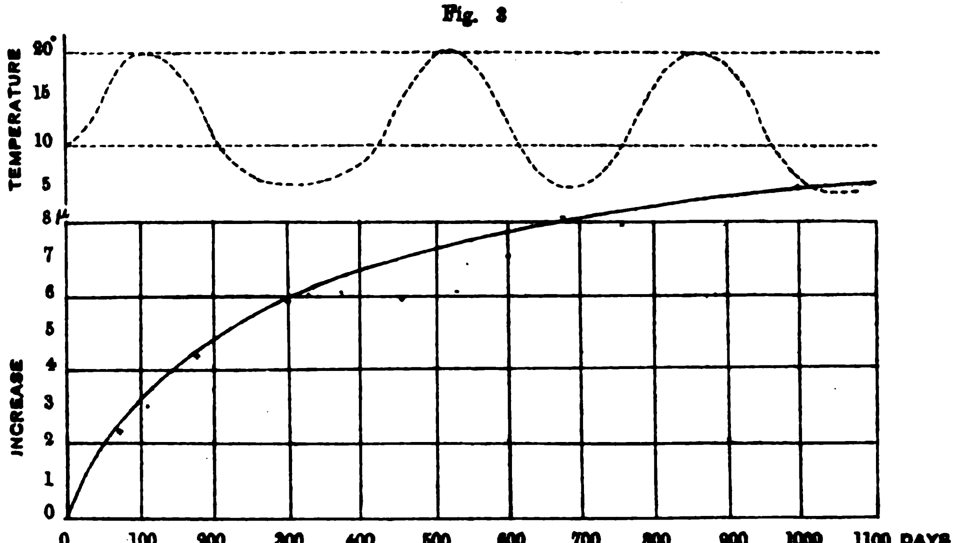


Fig. 3



VI

THEORY OF ELECTRIC PROJECTORS.

BY

ANDRÉ BLONDEL.

In giving a summary of the principles of photometry applicable to the study and construction of these instruments, a knowledge of all the fundamental laws and definitions of ordinary photometry and geometry is presupposed. The notation which will be generally employed is as follows:—

q = quantity of light (luminous energy divided by the time).

s = the surface illuminated.

E = intensity of illumination at a point: $E = \frac{dq}{ds}$.

i = illumination at a point with a given angle of emission ϵ: $i = \frac{dq}{ds \cos \epsilon}$.

I = luminous intensity in a given direction.

I_ϵ = " " at a great distance from instrument in a given direction.

l and l' = lengths or distances.

I

CHARACTER AND EFFECT OF PROJECTORS.

Optical instruments are composed of reflecting or refracting surfaces combined either to concentrate or to render parallel or to diffuse rays emitted by a source of light. I will only consider the first two cases.

Preliminary Remark.

When we consider only the general course of the rays, we are content, as we often do, to assume that the source of light is a geometrical point, as the reasoning is thereby much simplified.

When, however, we consider the photometric effect, we commit the gravest error in adhering to this theoretical conception.

Take for example the case of a lens, in the focus, F, of which a candle is placed; according to theory, we should obtain a pencil of parallel rays: practically, we obtain a pencil more or less divergent according to the dimensions of the flame (see Fig. 1).

It is the same with all optical instruments intended to render rays parallel, as these in general can always be considered equivalent to a more or less perfect lens. Even in the French electric light-houses, where the diameter of the incandescent parts of the pencils is sometimes as small as 5 mm., the pencil presents a divergence of more than 1°, and this figure is surpassed in the case of projectors. Even allowing that we still farther reduce the divergence, it can never entirely be removed, as it is impossible to reduce the source to a point. Consequently the pencils of electric projectors are, and will always be, conical, and not cylindrical.

Characteristic Elements of a Projector.

The useful effect of a projector is defined by two elements:—

1°. The luminous power, *i.e.*, the luminous intensity at a great distance, measured along the axis of the pencil. This determines the limit of the range at which the projector is able, according to the state of the atmosphere, to produce a sufficient intensity of illumination.

2°. The amplitude and the composition of the pencil which determine the extent of the usefully illuminated space.

1°. The Power.

Each point, *m*, of the lens (Fig. 1), receiving from the source a little conical pencil of rays emits round the horizontal, *mh*, a little pencil, also conical, whose minimum angle is practically of the order of magnitude of 1°. All the points of the lens playing a similar rôle, it becomes a secondary source of light presenting a certain intrinsic brightness like all sources of light, and consequently we can apply to it, in a perfectly legitimate manner, the notion of ***intensity at a great distance*** measured by the illumination.

It is necessary to remember that the ***great distance*** is entirely relative. If 3 m. is a great distance for a Carcel lamp, 300 m. will be equally a great distance for projectors whose diameters so far do not exceed 1·5 m.

Now it is a fact of experience, the theory of which will be given elsewhere, that, after a certain distance the illumination measured in the axis of a pencil by the observer varies in the ratio of the inverse square of the distance to the surface of the projector. There is nothing to hinder the application to that secondary source of the definition applicable to all sources of great dimensions, and to say with all professional engineers:—***The luminous intensity of an optical instrnment at a great distance in the axis of the pencil is the constant quantity to which the product of the intensity of illumination by the square of the distance rapidly attains***

$$P = l^2 E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is to say, consequently, the intensity of a naked light which would produce the same illumination at the same distance.

In France, to better characterize what is meant by an intensity at a great distance, the expression ***luminous power of an instrument*** strongly advocated by M. l'Inspecteur général Bourdelles, is employed in preference, but the expression ***intensity*** is none the less perfectly rigorous.

The distance at which the illumination begins to vary according to the law of squares is always sufficiently small relatively to the ordinary distances for which the instruments are used, that ***the law of variation at a closer distance does not present any practical interest.***

2°. The Amplitude and Composition of the Pencil.

The amplitude of the pencil at a great distance is measured by the angular opening, or better, by the tangent of that angle; it depends on the distance and on the transparency of the air, because the marginal rays, generally more feeble than the central ones, are extinguished first by atmospheric absorption. If the instrument is well made, the elementary pencils have all sensibly the same axes at a great distance so that the illumination is maximum and almost uniform in the solid angle embraced by the rays of minimum divergence. We can thus assign to the part of the rays which reaches the limit of the range, the minimum divergence of the elementary pencils. The maximum divergence has only interest at a short distance. But it is very important in practice that it may be as little as possible greater than the minimum divergence so as to have a pencil as homogeneous as possible.

If, instead of placing himself in the axis of the pencil, the observer places himself on a line, C*p*, (Fig. 2) drawn in an oblique direction, C representing the projector (which on a small scale figure is necessarily reduced to a point), he will experience an intensity, I_g, more feeble in general than on the axis, and one which will go on decreasing up to the edges CL, CL'. *

The law of distribution of intensity at a great distance can therefore be defined by a curve of intensities just as for sources of ordinary light. The only difference is that the curve is very much elongated, and contained in an angle much smaller than for ordinary sources: but that does not constitute a difference in principle. There are other sources, for example the electric arc, which illuminate only certain directions in space leaving others in obscurity.

The form of this curve not being of practical use, it is preferable to define the composition of the pencil by another curve (Fig. 3) representing the variation of intensity of illumination in a right section, a_1da_2, of this pencil. In this design the abscissæ are the distances of a point from the axis, and the ordinates the corresponding illumination. The form of the curve thus obtained depends essentially on the source of light employed. With oil lamps, the pencil is large and its intensity decreases in a progressive manner from the centre to the edges, following a law which, according to M. Allard, is almost parabolic. On the contrary, the electric arc gives a pencil with very small opening and nearly homogeneous, *i.e.*, one whose intensity only diminishes very little from the centre to the edges, where it is extinguished almost abruptly—as is shown in (Fig. 3)—in the case of a good pencil. That is easily explained, for the useful part of the arc is sensibly reduced to that of its crater, which has almost a uniform brightness.

The most useful part of the circle of illumination, a_1da_2 (Fig. 3), is the central spot b_1b_2, for it is there that the light is maximum. Experience shows that with absolute equality of illumination at the centre of a pencil, objects are so much the more visible as the extinguishing of the intensity on the edges is the more rapid, for the contrast between the illuminated object and the sombre space surrounding it which contributes to make it seen, is so much more pronounced. When the central zone is surrounded by a very large exterior zone of decreasing clearness the objects illuminated stand out less plainly on the background and are drowned in a luminous haze, produced by the dust and vesicles of the atmosphere illuminated by the edges of the pencil, which spoils the definition of objects illuminated by the central spot.

II

CALCULATION OF THE LUMINOUS POWER AND OF THE DIVERGENCE OF A PENCIL OF AN OPTICAL INSTRUMENT.

The calculation of the luminous maximum intensity (or power) of an instrument, from what precedes, resolves itself into that of the intensity of illumination produced at a point of the optical axis at a great distance.

As the basis of this research I will indicate a theorem, which I think is new, not having found it in any of the most recent works on optics, and which enables us to execute the calculations of geometrical photometry in a rigorous manner.†

Fundamental Theorem.—Let us consider any refracting surface, Σ, separating two isotropic media: let us suppose that any source of light, S, is placed on one side, and let us propose to calculate the intensity of illumination produced at any point, M, in space, this illumination being

* The pencil, at first fairly large, goes on getting smaller as the distance increases, while preserving its conical form with the point, C, as vertex.

† Verdet, who *en passant* has treated the question of the calculation of luminous intensity, only gives inexact or insufficient methods. Other physicists have in general confined their attention to perfect lenses.

measured on a given plane (Fig. 4). Let us apply at once the principle of the return of rays and trace all the rays issuing from M and falling on the surface, Σ. After refraction, in passing through this surface, these rays will take various directions and in general only a part of them, which I will call the *useful rays*, will reach the source, S. These rays, which will cut off on the surface, Σ, one or more portions, σ, more or less extended are evidently the only ones which, conversely, are able to contribute to the illumination of the point, M. All the part of the surface comprised outside σ does no work for the point, M, as one ascertains, for example, when one places oneself very near a projector or an electric light-house, the centre only of the mirror or of the lens appears bright, the rest being obscure.

Everything is now reduced to determining the illumination produced in M by each infinitely small element, $d\sigma$, of the surface and making the sum. The choice that I am going to make of a method of decomposing σ into elements depends on the consideration of focal lines. *Á priori*, the rays proceeding from the source, conjugate to the rays proceeding from the point, M, are normal to the same surface (in virtue of a theorem of Malus), and if from among them we separate an infinitely small pencil which cuts off on the said surface a little rectangle formed by four lines of curvature infinitely close, all the rays of this pencil pass on through two infinitely small right focal lines situated in planes at right angles (theorem of Sturm).

I will suppose that we decompose all the useful rays proceeding from the source into a series of pencils of this description which cut the surface, Σ, in a series of little corresponding elements forming a four-sided figure which I have not drawn.

Let us represent on a large scale in ABCD, (Fig. 5), one of the little elements, $d\sigma$, the pencil of rays proceeding from M and abutting on this element, the two focal lines, ab and cd, of the corresponding conjugate, pencil, and lastly the element, ds, cut by the pencil from the radiating surface of the source of light, S. (fig. 4).

Every point, p, of the element $d\sigma$ receives from ds a cone of rays of which it is the vertex, and the base of which is ds: this cone is refracted into another cone with the same vertex, and which contains the same flux of light. All the analogous cones corresponding to points of $d\sigma$ differ infinitely little among themselves, and can consequently be considered as all equal to two among them having for their common vertex B and containing a flux of light, dq: now if one traces at c and at M planes respectively normal to the axes of the incident and refracted pencils, which they cut in two elementary parallelograms $d\epsilon = cfgh$ and $d\epsilon' = Meqo$, the illumination produced on these planes by the two cones proceeding from B will be

$$\frac{dq}{d\epsilon} \text{ and } \frac{dq}{d\epsilon'}.$$

The sums of the intensities of illumination at M and at c due to all the little equal cones will have, therefore, for their ratio the expression.

$$\frac{de'}{de} = \frac{d\epsilon}{d\epsilon'}$$

The element, $cfgh$, which is a rectangle, (in virtue of the theorems of Malus and Sturm) has for its surface—

$$d\epsilon = l_1^2 \, d\omega \, d\omega_1$$

$d\omega$ and $d\omega_1$ being the angles of the faces of the elementary cones, and l_1 the distance of the focal line cd, from B.

Similarly $\quad d\epsilon' = l'^2 \, d\omega' d\omega_1' \sin\theta'.$

$$\text{whence } \frac{de'}{de} = \frac{l'^2 \, d\omega \, d\omega'}{l'^2 \, d\omega' \, d\omega_1' \sin\theta'}.$$

Now we can, on the other hand, write by a well known formula

$$de = i\, d\lambda.\, d\lambda_0$$

where $d\lambda_0$ and $d\lambda$ are the angles under which are seen the two sides of the elementary parallelogram, $d\sigma$, at c.

Finally, we can, by calling l the distance of the second focal line, ab, from B, and λ_1, the angle which one of the sides subtends at a, replace the product, $l_1\, d\, \lambda_0$, by $l\, d\, \lambda_1$, so that we finally get

$$de' = i\, d\lambda\, d\lambda_1 \frac{l\, d\omega}{l'\, d\omega'} \cdot \frac{l_1\, d\omega_1}{l'\, d\omega'_1} \cdot \frac{1}{\sin\theta'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This formula constitutes the theorem which may be enunciated thus:—

The elementary intensity of illumination produced at a point, M, by an element of the surface, $d\sigma$, cut by an incident pencil having for its base on the wave surface a rectangle of lines of curvature, is equal, not considering the coefficient of loss in passage, to the product of the intrinsic brightness, at the point where the corresponding rays leave the source, by the focal angles, $d\lambda$, $d\lambda_1$, of the incident pencil and a factor $\frac{l\, d\omega}{l'\, d\omega'} \cdot \frac{l_1\, d\omega_1}{l'\, d\omega_1'} \cdot \frac{1}{\sin\theta'}$: the ratio of divergence $\frac{d\omega}{d\omega'}$ and $\frac{d\omega_1}{d\omega_1'}$, being measured, respectively, in two planes at right angles passing through the axis of the little pencil and through its two focal lines and in the two corresponding refracted planes passing through the point, M, and making with each other an angle, θ', which should be calculated in each case; l', l and l_1 representing, on the other hand, the distances of the element, $d\sigma$, to the point, M, and to the two corresponding focal lines

To get the total illumination at M it will suffice to integrate the expression for all the useful rays, taking account of the inclination, ϵ, to the plane, P, of rays abutting at M. We will, therefore, have—

$$e' = \iint de' \cos\epsilon = \iint ki\, d\lambda\, d\lambda_1 \frac{l\, d\omega}{l'\, d\omega'} \cdot \frac{l_1\, d\omega_1}{l'\, d\omega_1'} \cdot \frac{\cos\epsilon}{\sin\theta'} \quad — \quad . \quad . \quad . \quad (3)$$

k being the coefficient of loss by refraction in the passage through the surface, Σ.

In the case of a completely general system comprising several successive refracting surfaces we have only to apply the same method for the trace of the useful rays and to calculate at each surface of passage the two factors of concentration. This procedure, extremely complicated in appearance, is luckily much simplified in practice, thanks to circumstances and to the employment of some justifiable hypotheses.

Utilisation of the Theorem.—In practice we ought to try and reduce all the successive refractions and reflections to a single refraction at a surface, real or imaginary, which will then be considered as the surface of emission.*

In the most general case such a surface does not exist, for the conjugate rays entering and leaving, not being in the same plane, cannot cut.

But there are two cases in which such a surface can be found:—

1°.—When all the optical system is centred, *i.e.*, one of revolution round an axis and when we seek the intensity of illumination at a point in this axis (the case of projectors and of annular lenses). Then the rays entering and leaving cut each other two and two and their consideration is simple.

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* One would be tempted to apply for this purpose the *theorem of Gergonne* given in all treatises on optics in the following form:—"The effect of any number of reflections and refractions can be replaced by the effect of a single refraction taking place according to a constant index chosen arbitrarily. (*Annales de Mathématiques*, t XIV, p. 129, 1828. See also Mascart *Optique*, t I, p. 62.)

But this statement is absolutely incorrect and has already caused many errors. Verdet himself (*Optique Physique*, t I) accepts it in proposing its application to Geometrical photometry.

In reality the pencils are equivalent as far as the form of the undulation is concerned, but not in regard to the intensity of the vibratory movement at each point of the wave.

2.°—When we are treating of a reflecting or refracting system so that we can consider it without great error as a surface without thickness. This hypothesis can be admitted in most practical applications, in particular for projectors.

On that single refracting surface, Σ, we shall have to calculate at each point the deviations of the rays and the coefficients of divergence $\frac{d\omega}{d\omega'} \cdot \frac{d\omega_1}{d\omega_1}$, according to the actual construction of the lens or mirror.

Case where the instrument has a focus.—In most cases in practice when we employ optically centred instruments the pencil of rays, emanating from the point, M, situated on the axis at a great distance, has for its conjugate a pencil of which all the focal lines sensibly pass through a mathematical point, F, called the focus of the instrument and around which the luminous surface is placed. The general formula then takes a more simple form, for $l_1 = l$ and the product, $d\lambda . d\lambda_1$, is equal to the solid angle, $d\Omega$, of the pencil of rays emitted from the focus, F. We have therefore

$$de' = ki \, d\Omega \frac{l^2}{l'^2} \frac{d\omega \, . \, d\omega_1}{d\omega' . \, d\omega_1' \, \sin \theta'}.$$

More generally we can demonstrate with a little labour by means of spherical trigonometry that the incident pencil can be decomposed into elementary pencils chosen in any way * on the condition of writing by analogy,

$$de' = ki \, d\Omega \frac{l^2}{l'^2}, \frac{d\omega \, . \, d\omega_1}{d\omega' . \, d\omega_1'}, \frac{\sin \theta}{\sin \theta'}.$$

In the theoretic case of a single refracting surface considered in the fundamental theorem, the last ratio which figures in that formula is a constant whose value is $n^2 \frac{\cos \delta'}{\cos \delta}$; δ and δ' being the angles which the rays make with the normal on entry and exit at the surface, σ, and n the ratio of the indices of the extreme media.

Case of projectors.—The general theorem is applied easily and with great simplification to projectors.

1.°—All projectors being ones of revolution round their optical axis, the power will be calculated by cutting up the pencil of incident rays into elementary pencils by means of meridian planes and cones of revolution having the focus for vertex † as shown in Fig. 6. In these conditions the coefficient of divergence $\frac{dw_1}{dw_1'}$, in the direction perpendicular to the meridian plane is equal to unity, $\theta' = \frac{\pi}{2}$ and we simply have

$$de' = ki d\Omega \frac{d\omega}{d\omega'} \cdot \frac{l^2}{l'^2} \quad \ldots \ldots \quad (4).$$

*That is otherwise evident since the surface of undulation is a sphere on which the direction of the lines of curvature is arbitrary.

†If the instrument presents aberration for rays parallel to the axis, or if we wish to calculate the illumination on the axis at a short distance (Fig. 8) we have no longer the right to consider a single focus, F. The focus is replaced according to the general theorem by the two focal lines: these are here the optical axis itself and a circle perpendicular constituting a parallel of the caustic surface, the envelope of the conjugate rays of the point, M, where the illumination is required.

The expression for de' is therefore

$$de' = ki \frac{d\omega}{d\omega'} \quad \frac{l_1 \, l}{l'^2} \, . \, d\lambda \, d\lambda_1$$

l and l_1 being the two focal lengths reckoned to the caustic and to the axis: and in consequence of this calling δ and δ' the angles between the normal at $d\sigma$ and the incident and departing rays.

$$E' = \int \frac{ki \, d\sigma \cos \delta}{l'^2} \quad \frac{dw}{dw'} \quad \cos \epsilon$$

ϵ being the angle which the ray passing through the illuminated point, M, makes with the axis. At a great distance $\cos \epsilon = 1$ and the formula is the same as without aberration.

When the observer is at a great distance the rays can be considered parallel to the optical axis. If ds be the section of an emergent pencil by a plane perpendicular to the optic axis, that is to say, the projection of $d\sigma$ on that plane and δ and δ', the angles between the normal at $d\sigma$ and the incident and emergent rays, we can write, remembering that $l'^2 d\Omega = d\sigma \cos\delta$ and $ds = d\sigma \cos\delta'$,

$$E' = \frac{1}{l'^2}\int \frac{d\,s \cos\delta}{\cos.\delta'}\,\frac{d\omega}{d\omega'}\,ki \;=\; \frac{1}{l'^2}\int kuids \quad . \quad . \quad . \quad . \; (5)$$

that is to say that :—

at a great distance every projector will behave like a plane circular disc having for its intrinsic brightness at each point (kui)

$$\text{where } u = \left(\frac{\cos.\delta}{\cos.\delta'}\cdot\frac{d\omega}{d\omega'}\right) \quad . \quad . \quad . \quad . \quad . \quad . \; (6)$$

The factor between brackets measures therefore for each zone of the surface, Σ, the *optical effect of the projector* and gives from this point of view an idea of the value of the instrument.

The apparent brightness, kui, of each point of the surface can be determined experimentally;* but I will proceed to give here the calculation of the coefficient, u. The coefficient, k, is determined by Fresnel's formula of reflection and by the law of absorption of glass.†

2_o.—The divergence of each little pencil in the meridian plane and in the direction perpendicular are obtained approximately, thanks to the small dimensions of the source of light, by multiplying by the coefficients of divergence $\frac{d\omega}{d\omega'}$, and $\frac{d\omega_1}{d\omega_1'}$ -- (which is equal to unity), the two angles under which that source is seen from the corresponding point of the surface, Σ. I will not enter, for want of space, into the details of this calculation which the reader will easily make for each projector. I will indicate only for the most important types the divergence, α', obtained by supposing (Fig. 7) the arc reduced to a luminous circular crater, d, completely exposed and perpendicular to the axis‡.

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* For example by photographing the illuminated instrument by the aid of a tele-objective at a great distance, or by examining it from a distance with a microphotometer, or by studying the horizontal intensity produced by each point of the apparatus (the rest being stopped out) on a screen H (Fig. 7), in which case the pencil can be enlarged by means of a lens L.

† Calling x the fraction of the light reflected at the surface separating two media of which the second has an index n with regard to the first so that $\sin i = n \sin r$, the formula of Fresnel is, as is well known

$$x = \tfrac{1}{2}\,\frac{\sin^2(i-r)}{\sin^2(i+r)} + \tfrac{1}{2}\,\frac{\tan^2(i-r)}{\tan^2(i+r)}.$$

As for absorption it has the effect of enfeebling the intensity I of the light according to an exponential law, a function of the thickness (a) traversed, *viz.*

$$I = I_0\, m^a$$

m being an experimental coefficient. According to the measures of M. J. Rey on commercial glasses, we can put $m = 0{\cdot}985$, a being measured in centimetres. In practice we can replace the exponential formula by the following—

$$I = I_0\,(1 - ma)$$

which gives almost exactly the same values so long as a is less than 10 centimetres.

It is difficult to know exactly the coefficient of reflection of silvering, for very discordant figures have been given up till now; but it is certain that the reflection on silvering in contact with glass, when it is well made, absorbs very little light, 9 per cent. according to Herschel. I have adopted 10 percent. in the remainder of this work.

‡ It has been known for a long time that the lighting effect of an arc lamp with continuous current proceeds almost entirely from the crater of the positive carbon which furnishes 85% of the total light, (Industrial photometry of Palaz). The recent and curious researches of Mr. Trotter (Inst. of Electrical Engineers, England, May 1892) have given a particularly neat and interesting demonstration of this fact at the same time that they have established, for the first time, that a crater behaves exactly like a luminous circular disc uniformly brilliant and emitting light according to Lambert's law, and that every thing takes place for an observer on the side of the negative pencil as if this latter had no luminous power. This remark of which I have recently indicated another application (On the Electric Light of Light-houses, International Maritime Congress, 1893) is to-day interpreted very well by the theory of the arc proposed by Mr. S. P. Thompson, established by M. Violle, verified and completed by a recent work on the arc with continuous current (*The Electrician*, December 1893). It is, therefore, perfectly legitimate to suppose the theoretical arc reduced to a brilliant circular disc. The effect which the negative pencil has in shutting off the light will be considered later on.

This divergence α' is the minimum divergence of each elementary pencil on an elliptical base; it has for a general expression in the case considered supposing d very small

$$\alpha'=\alpha.\frac{d\omega'}{d\omega}=\frac{d\omega'}{d\omega}.\frac{d\cos\gamma}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

l being the length of the ray, F P, and γ the angle which it makes with the normal to the crater; α' being proportional to d for each given instrument it can be defined as far as the divergence is concerned by the values of the ratio

$$w=\frac{\alpha'}{d}=\frac{d\omega'}{d\omega}.\frac{\cos\gamma}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

at each point of the surface. I will indicate in each particular case the definitive expression w and I will trace its curve as a function of the radius of each parallel of the instrument. In the actual types employed w is maximum at the centre and minimum at the edges of the instrument. In calculating these values we have at once the ratio of the diameter of the central spot and that of the whole section of the pencil.

Variation of the intensity of illumination with the distance.—We are now able to study with more precision the variation of the intensity of illumination with the distance. Let us take the case of a centred instrument having a fictitious refracting surface, Σ, (Fig. 8) and a focus, F, around which the source, S, is placed. The rays conjugate to those which pass through M envelope a caustic surface of revolution having a meridian, COC′, which is distorted when the point, M, is displaced.

When the point, M, is near the instrument the central rays alone carry light to it, the others not meeting the flame. In proportion as M moves further away the point, O, approaches the focus, F, and at the same time the caustic is reduced; the number of useful rays increases and the instrument which only appeared brilliant at the centre is illuminated little by little to the edges.* There comes a time when the caustic is reduced to a very small surface which is no further sensibly distorted and which should become a point if the instrument were absolutely free from aberration.

From this time as the angle of incidencc, ϵ, is very small and $\cos\epsilon$ sensibly equal to unity, the formula (5) shows that E' is simply proportional to $\frac{1}{l'^2}$ and that one can determine in a legitimate manner the luminous intensity or the power of the instrument.

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* The useful surface for each point of the optic axis is easily determined when we have calculated for each parallel of the mirror the divergence, α', for which the expression is given in the preceding paragraph. In fact if we turn to Fig. 7 we see that the lower ray of the little pencil, α', meets the optic axis at a distance

$$l'=\frac{r}{\frac{\alpha'}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

always supposing the angle, α', to be so small that it may be considered equal to its tangent (otherwise α' must be replaced by tan.α'). Consequently if we place the eye on the optic axis at a distance l' the illuminated surface of the projector will appear limited by the parallel of radius r. When we trace the curve of α' or of w as a function of r as before explained, we have immediately by formula (9) the law of variation of l' as a function of r. We could deduce the illumination at each distance by the formula given before

$$E'=\int_0^r \frac{ki\,d\sigma\cos\delta}{l'^2}.\frac{d\omega}{d\omega'}\cos\epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

without knowing the focal lengths l & l_1 or the caustic conjugate to the point, M. This research is in general too laborious to be touched upon here, and besides, as I have said, it does not present any interest.

The minimum distance where that experimental determination can take place ought to be the greater as the source of light is smaller and of a less uniform brightness * Calculation† and experiment show that for ordinary lenses a distance equal to 100 times the focal length is in general sufficient where oil lamps with large flames‡ are employed; but for electric projectors it is necessary to place oneself at a distance of at least 500 metres to have a measure worthy of confidence making the corresponding correction for atmospheric absorption.§

The luminous intensity so defined avoids criticism and there is generally no interest in knowing the values of the illumination at a shorter distance. We must not forget in fact that the range of utilization at present reaches 8 or 10 kilometers for large projectors and for electric flash lights in clear weather has no other limit than the geographical one, and that in the immediate neighbourhood of the instrument we have always more light than we want.

III.

APPLICATION TO DIVERS CLASSES OF PROJECTORS.

These general principles being laid down, I can now apply them to various classes of projectors, that is to say, to lenses in echelon, to parabolic mirrors and to Mangin's mirrors. In this study, I will consider the optical instrument only, *i.e.*, unembarrassed by simple glass screens or by diverging screens and neglecting the occultation produced by the uprights of the lamp. The effect of a glass screen is to produce a loss of about 0·9: the effect of the diverging screen is to display the pencil horizontally in augmenting its amplitude at the expense of its intensity; at the same time the diverging glasses absorb a very important quantity of light. The theory of this accessory part of projectors does not present any difficulty, it is useless to speak of it here. Similarly, I will leave aside the new system of torique projectors of Messrs. Sautter-Harlé which bears the same relation to an ordinary Mangin projector that the fixed light does to the flash light in the matter of light-houses. Otherwise its theory is the simple adaptation of my general theory to a plane figure.|| I will indicate in passing, on account of their theoretical interest, the formulæ in regard to metallic

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* M. Allard has indicated another method, unfortunately a little laborious, which permits the measures to be made in a laboratory: the light emitted by the several parts of the surface of the instrument is studied separately; this method avoids the errors due to atmospheric absorption.

† We can determine very easily the minimum distance at which the observer should be placed on the optic axis so as to see the whole projector illuminated. Calling $2r_0$ the opening of the projector and α'_0 the divergence of the pencil whose vertex P (Fig. 7) is on the edge of the projector, that distance is evidently

$$l'_0 = \frac{2r_0}{\alpha'_0} = \frac{2r_0}{w_0 d}$$

d being the diameter of the crater and w_0 the factor of divergence at the edge which is easily determined for each projector. Applying this formula to a projector of 0·900 metres and focus 1·015 metres illuminated by a crater of 0·01 metres. The table on page 41, gives $w_0 = 0·86$ whence

$$l'_0 = \frac{·9}{·01 \times ·86} = 105 \text{ metres}$$

but this distance is too small to make a good measure, for, one would see the whole projector luminous but would be on the edge of the extreme little pencils. It is therefore well to place oneself at least 4 times further off to be sure of being in the middle of these pencils.

‡ Concerning light-houses and their effect at a short distance see the Memoir of M. Bourdelles "On the Luminous Power of Light-house Apparatus." mentioned further on.

§ This is what one does in all trials of projectors so that the measures are in perfect conformity with the definitions.

|| The formula (5) is replaced by the following

$$E' = \frac{d}{l'^2} \int kui\, dy$$

d being the diameter of the crater and dy the projection of an element of the meridian curve on the axis of the torus.

mirrors and perfect lenses to link them to the general theory and not to omit any of the instruments which can be employed for the projection of light. I take the case of the arc with continuous current; it is unnecessary to say that the formulæ are general and applicable to other sources of light.

1° *Lenses.*—Let BCK (Fig. 9) be the meridional section of a lens composed of a single piece of glass. Let δ, r, r', β be the successive angles of incidence and refraction of a ray ABCD at the two surfaces. Let us give to the ray, A B, an infinitely small deviation, $d\omega$, in the plane of incidence; it is easily seen if a the length of B C and R the radius of curvature of the face, CK, at C, that the corresponding deviation of the refracted ray has the value

$$d\omega' = d\omega.\ \frac{\cos r'}{\cos r}.\ \frac{\cos \delta}{\cos \beta} \left(\frac{a}{R \cos r'} - 1\ ^{*}\right).$$

Practically we can, at least in an approximate calculation, neglect the thickness of the lens, a, compared with its radius of curvature and write the absolute value

$$\frac{d\omega}{d\omega'} = \frac{\cos r}{\cos r'}.\ \frac{\cos \beta}{\cos \delta} \quad \dots \dots \dots \quad (11)$$

On the same hypothesis the surface, Σ, sensibly becomes the anterior face of the lens, and it is consequently sufficient to consider this as the surface of emission of light.

The most elementary application is that relating to a theoretically perfect lens whose conjugate foci satisfy the known relation

$$\frac{1}{l} + \frac{1}{l'} = \frac{1}{f} \quad \dots \dots \dots \quad (12)$$

Let us seek (Fig. 10) the illumination produced at a point, M, in the axis by an indefinite luminous surface, L, of which the brightness, i, is supposed to follow Lambert's Law, *i.e.*, it is constant under all incidences.

Let M′ be the conjugate of M satisfying the relation (12). In the case of theoretical lenses the opening is supposed so small that $\delta = nr$ and $\beta = nr'$. We find thus, calling S the total surface of the lens

$$e' = \int de' = \frac{ki\,S}{l'^2} \quad \dots \dots \dots \quad (13)$$

that is to say that apart from the coefficient of loss, k, *the lens behaves at all distances like a source of light having the same intrinsic brilliancy as the source of light*, L.

* Translator's note:—

Let O be the centre of curvature at C

then angle COC′ $= r + dr + r' + dr' - r - r'$

$= dr + dr'$,

also C′N′ = CC′ $\cos r'$ when C′N′ is at right angle to BC.

$\therefore a\,dr = CC' \cos r' = R \cos r' \times$ angle COC

$$\therefore \left(\frac{a}{R \cos r'} - 1\right) dr = dr'.$$

Now $\sin \delta = n \sin r$

$\sin \beta = n \sin r'$

$$\therefore \frac{\cos\delta}{\cos\beta}\ \frac{d\omega}{d\omega'} = \frac{\cos r}{\cos r'}.\frac{dr}{dr'}$$

$$\therefore d\omega' = d\omega \frac{\cos r'}{\cos r}\ \frac{\cos\delta}{\cos\beta}.\ \left(\frac{a}{R \cos r'} - 1\right)$$

We thus obtain a known result which has been indicated by M. Cornu and by Colonel Mangin,* but its demonstration has habitually been given only by the consideration of images. We see, on the contrary, that the source, L, can be placed anywhere provided it is cut by all the rays.

When it is very small it is necessary to place it at the point, M, itself, otherwise in the formula (13), S will represent only the useful part of the surface, which will go on increasing with the distance. Starting from the time when the entire surface is illuminated, we can define the intensity at a great distance, or the power, by the formula—

$$P = l'^2 e' = ki\,S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

When the opening of a simple lens becomes somewhat considerable, the spherical aberration and the thickness become enormous. It is to reduce these that Fresnel has contrived the lenses in echelon whose profile comprises (Figs. 11 and 12) a series of elements having each for its anterior profile a vertical straight line, and for its posterior profile a curve so calculated as to send horizontally the rays proceeding from the focus, F, of the instrument.

Let ABC be the section on a large scale of one of these elements, and SABD one of the rays. Calculating from this figure the angles β, r and r', and substituting these values in the formula (6), we find,†

$$\frac{d\omega}{d\omega'} = \frac{1}{\cos\delta}\,\frac{\sqrt{n^2 - \sin^2\delta}\,(n^2 + 1 - 2\sqrt{n^2 - \sin^2\delta})}{n^2 - \sqrt{n^2 - \sin^2\delta}}$$

a value which admits of a purely analytical integration in the form of a series.

It is better to make the integration graphically by determining the curve of the product, kui, as a function of the radius, r, of each parallel.

One can calculate by this formula the optic effect, $\frac{d\omega}{d\omega'}\cos\delta$ (since $\beta = 0$) and compute the following table. The coefficient of loss, k, has been calculated, taking count of reflections and of absorption; the loss corresponding to this is very nearly equal for small thicknesses to $0{\cdot}015 \times a$, where a is the thickness traversed.

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* M. Cornu, *Annales de l'Observatoiré*, t. xiii, and M. Mangin, *Memorial de l'Officer, du Gènie*. t. xxiii. This property of lenses can be very usefully employed to determine the brightness of a source of light by projecting the image on a photometer, the surface, S, and the coefficient, k, of the lens are determined directly. We can apply it in particular to the standard arc making use of a lens instead of a diaphragm. M. Cornu has eliminated the measurement of the constants, k, by employing two lenses equally provided with iris diaphragms (*see* Photometric studies of M. Cornu in the Journal of Physics).

† Translator's note—

$$\sin\delta = n \sin r', \quad \sin\beta = n \sin r \text{ and } \beta = r + r'$$

$$\therefore \sin r \cos r' + \cos r \sin r' = n \sin r$$

$$\therefore \tan r = \frac{\sin r'}{n - \cos r'}$$

$$\therefore \sec^2 r\, dr = \frac{(n - \cos r')\cos r' - \sin^2 r'}{(n - \cos r')^2}\,dr' = \frac{n\cos r' - 1}{(n - \cos r')^2}\,dr'$$

$$\text{and } \sec^2 r = 1 + \tan^2 r = \frac{n^2 - 2n\cos r' + 1}{(n - \cos r')^2}$$

$$\therefore dr = \frac{n\cos r' - 1}{n^2 - 2n\cos r' + 1}\,dr'$$

$$\therefore d\omega' = d\beta = dr + dr' = \frac{n^2 - n\cos r'}{n^2 - 2n\cos r' + 1}\,dr'$$

$$\text{also } d\omega\cos\delta = d\delta\cos\delta = n\cos r'\,dr'.$$

$$\therefore \frac{d\omega\cos\delta}{d\omega'} = \frac{n\cos r'\,(n^2 - 2n\cos r' + 1)}{n^2 - n\cos r'}$$

$$\text{and } \sqrt{n^2 - \sin^2\delta} = n\cos r'$$

Lenses in echelon; focal length = 1 metre.

Radius of each zone in metres.	Coefficient of optic effect u.	Coefficient of transmission k approximately.	Factor proportional to the divergence $w = \frac{\cos^3 \delta}{u}$
0·00	1·00	—0·88	0·99
·10	·995	·88	·98
·20	·99	·88	·95
·30	·97	·87	·89
·40	·95	·86	·84
·50	·92	·85	·78
·60	·89	·85	·72
·70	·86	·84	·66

Calculation of the coefficient of transmission, k, for lenses in echelon.

Angles.			Mean thickness of glass of corresponding annulus estimated according to existing patterns in centimetres.	Coefficients of transmission.			
Of incidence. δ	Of emergence. δ'			1st refraction, Fresnel's formula.	Absorption.	2nd refraction, Fresnel's formula.	Resulting coefficient k.
°	°	'					
0	0	0	2·5	0·995	0·965	0·955	0·88
10	18	10	2	·955	·970	·955	·88
20	34	22	,,	·955	·970	·95	·88
30	38	10	2·5	·95	·965	·95	·88
40	47	23	,,	·95	·960	·94	·86
50	66	20	3	·935	·955	·88	·80
60	72	30	,,	·911	·950	·81	·70
70	76	50	3·5	·829	·945	·72	·56

We see that up to 40°, which is seldom passed in practice, the total coefficient, k, can be taken as almost constant and equal to 0·88, a figure which must be reduced if we take into account the loss of light produced by the joints. We can take 0·85 as a very good practical figure.

Above 50° the losses by reflection become enormous and the coefficient, k, diminishes so rapidly that the employment of lenses in echelon of large opening should be abandoned. After 45° it is usually better to adopt, in place of dioptric rings, catadioptric ones, of which we will speak further on, for they have a coefficient, k, $>$0·70 generally. The expression for the coefficient of divergence, w, is, as is easy to show,* calling f the focal length—

$$w = \frac{\cos^2 \delta}{f} \cdot \frac{d\omega'}{d\omega} = \frac{\cos^3 \delta}{fu}$$

* Translators' note—

$$w = \frac{\cos \gamma}{l} \frac{d\omega'}{d\omega} = \frac{\cos \delta}{l} \cdot \frac{d\omega'}{d\omega}$$

$$= \frac{\cos^2 \delta}{f} \cdot \frac{d\omega'}{d\omega}$$

$$= \frac{\cos^3 \delta}{f} \cdot \frac{d\omega'}{d\omega \cos \delta}$$

$$= \frac{\cos^3 \delta}{fu}.$$

I have indicated the values of the ratio,

$$w = \frac{\cos^3 \delta}{u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

from which the calculation of the values of w for any focus is easily made. Fig. 13, for example shows the curve, w, in the case of a focus, f, = 1 metre.

2°. *Metallic mirrors.*—The angle of reflection being equal to the angle of incidence, we have $\delta' = \delta$, $d\omega = d\omega'$, and in consequence at a great distance in the general case when there is aberration,

$$E = \frac{1}{l'^2} \int kids,$$

ds being always the section of the emergent pencil. The two focal lines will be mf, and mh, measured from the mirror to the caustic and to the optic axis (Fig. 15). The aberration is enormous in large spherical mirrors.

In spherical mirrors of small opening the caustic is reduced sensibly to a point. It is the same with parabolic mirrors which are rigorously aplanatic whatever the opening may be, and their formula is the same—

$$E = \frac{1}{l'^2} \int kids \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

3°. *Thin silvered glass mirrors.*—To avoid the sulphuration of silver exposed to the air, the metal is replaced by thin silvered glass with parallel faces: thanks to the small thickness (8 to 12 mm. in mirrors of 0·90 m.) the effect produced is the same as for metal mirrors except that the coefficient, k, is raised (0·80 to 0·85).

It is only of late years that they have succeeded in making for commercial purposes parabolic mirrors in silvered glass (Schuckert in Germany, Breguet in France). We see (Fig. 16) that with respect to the principal rays, M N, the aberrations of the parasite rays, mn, $m'n'$, reflected at the anterior surface or twice reflected, decrease with the thickness of the glass and the opening of the mirror. We can admit in general that they assist in part of the lighting, whence $k = 0·80$ to 0·85 about; and that all takes place at each point as in a mirror with parallel faces, whence $\frac{d\omega}{d\omega'} = 1$.

Up to the present no one has made parabolic mirrors in thick glass: their computation would present complications similar to those of Mangin's projectors.

The divergence is characterized as we have seen by the ratio, w, defined before (page 34). The formula (8) becomes here—

$$w = \frac{\cos \gamma}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

γ being the angle at the focus, and l the length of the incident ray. If we call f the focal length of the parabola and x and y the co-ordinates of a point of the mirror, we know that—

$$y^2 = 4fx \text{ and } l = f + x.$$

therefore

$$w = \frac{f-x}{l^3} = \frac{f-x}{(f+x)^3} = 4f. \frac{4f^2 - y^2}{(4f^2 + y^2)^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Such is the expression of the divergence as a function of the focal length and of the ordinate of each point of the mirror. From this expression I have calculated the factors of divergence for three parabolic mirrors of 0·900 m. in diameter having foci equal, respectively, to 1·015, 0·645 and 0·34. The figures obtained have served to construct the three curves, w_0, w_1, w_2 of Fig. 21, and have been reproduced for the first mirror in the table of page 41. We shall see further on the deductions to be drawn from this comparison. It may be remarked that by (18) there is for a mirror of given opening $2y_0$ a value of f which renders maximum the divergence of rays emanating from the edge of the projector and, in consequence, the diameter of the central spot. It is obtained by calculating the maximum value of w for $y=y_0$, and is found to be,

$$f=2y_0\frac{1+\sqrt{2}}{4}=0{\cdot}6\times 2y_0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Such is the focal length which for each projector gives to the central part of the pencil the greatest amplitude. The curves of Fig. 25 calculated by the formula (18) show how the section of a pencil as a function of the focal length varies.

4°. *Colonel Mangin's refracting mirrors.*—(Figs. 17 and 18.)—Let there be a mirror with the anterior face a sphere with centre, O, and let F be the point chosen for the focus. There evidently always exists a form of posterior surface such that the rays arriving parallel to the axis after having been refracted, reflected and again refracted, may be sent exactly to the point, F. For each position of the point, F, the meridian curve can be determined analytically or graphically by calculation generally very laborious. M. Mangin has found that one can, by conveniently choosing the point, F, replace, with a great degree of approximation, this meridian by a circle whose radius he has calculated. Without entering into the details of the calculation, the very remarkable result obtained by M. Mangin is the almost perfect aplanatism of the mirror.*

Let FMPG be a ray from the focus, F. The surface, Σ, is the locus of the intersections of the incident and emergent rays. The ratio, $\frac{d\omega}{d\omega'}$, is calculated on the other hand on the figure by giving to the ray, FM, a little rotation round the point, m, or, which is sensibly the same, round the point, M.

If we call a and a' the lengths of PM and PG, and R and R' the radii of curvature of the anterior and posterior faces, δ the angle OMN, β the angle of reflection, we have to the 2nd order nearly,

$$\frac{d\omega'}{d\omega}=\frac{\cos r'}{\cos r}\cdot\frac{\cos\delta}{\cos\delta'}\cdot\left(\frac{a}{R'\cos\beta}+\frac{a+a'}{R\cos r'}-1\right)$$

In the elementary photometric calculation such as I have presented here, we can consider $\frac{a}{R'\cos\beta}$ and $\frac{a+a'}{R\cos r'}$ as negligible in comparison with unity, *i.e.*, can neglect the thickness of the glass and in consequence suppose the surface, Σ, in coincidence with the anterior surface, and write, as for thin lenses†,

$$\frac{d\omega}{d\omega}=\frac{\cos r}{\cos r'}\cdot\frac{\cos\delta'}{\cos\delta} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Measuring, then, all the angles δ, r, r', δ' at the same point, M, of the anterior surface of the mirror, we find from Fig. 18,

$$\tan.\delta=\frac{b\sin\delta'}{R-b\cos\delta'} \text{ when O F} = b.$$

$$u=\frac{d\omega}{d\omega'}\cdot\frac{\cos\delta}{\cos\delta'}=\frac{d\omega}{d\omega'}\sqrt{\frac{n^2-\sin^2\delta}{n^2-\sin^2\delta'}}$$

* For example, the calculated aberration of a mirror of 0·60 m. is 0·3 m m., and the measured aberration does not exceed 0·5 mm.

† In spite of these simplifications the calculation is sufficiently exact since the dimensions of the source are in general so great as to render negligible the effect of small aberrations thus fictitiously introduced into the mirror.

whence

$$\frac{d\omega}{d\omega'} = \frac{\cos\delta'}{\sqrt{n^2 - \sin^2\delta'}} \cdot \frac{\sqrt{n^2\left[R^2+b^2-2\,b\,R\cos\delta'\right]-(b\sin\delta')^2}}{R-b\cos\delta'}.*$$

This expression is too complicated for simple integration; but in calculating u for different increasing values of δ' we can make a graphic integration or a calculation by zones. I give here the values obtained in two Mangin projectors of 0·900 m., old and new patterns constructed by Sautter-Harlé.

MANGIN'S MIRRORS OF 0·900 M., AND FOCAL LENGTHS 1·015 M., AND 0·645 M.

Radius of each zone in metres.	Co-efficient of the optical effect u.		Co-efficient of approximate transmission k.		Factor proportional to the divergence $w=\left(\frac{\cos\gamma}{l}\cdot\frac{d\omega'}{d\omega}\right)$		Value of the same ratio for a paraboloid of the same focal length.	
	1st.	2nd.	1st.	2nd.	1st.	2nd.	1st.	2nd.
0·00	1·00	1·00	0·80	0·80	0·99	1·55	0·99	1·55
0·10	1·00	1·00	„	„	0·98	1·54	0·98	1·53
0·20	1·005	1·01	„	„	0·975	1·51	0·97	1·45
0·30	1·01	1·03	0·76	0·75	0·965	1·45	0·94	1·33
0·40	1·02	1·06	„	„	0·95	1·36	0·89	1·17
0·42	1·02	1·07	„	„	0·945	1·34	0·88	1·14
0·45	„	„	0·72	0·70	„	„	0·86	1·09

The calculation of the co-efficient of transmission has been made taking count of the absorption 0·015 per centimetre of glass traversed, of the loss, then, of the reflection on the film of silver which I have taken equal to 10 per cent., and, finally, of the losses by reflection at incidence and emergence.†

* Translator's note.

$$\sin\delta = n\sin r \qquad \sin\delta' = n\sin r'$$

$$\therefore\ n^2-\sin^2\delta = n^2\cos^2 r \qquad n^2-\sin^2\delta' = n^2\cos^2 r'$$

$$\therefore\ \frac{d\omega}{d\omega'} = \frac{\cos\delta'}{\cos r'}\cdot\frac{\cos r}{\cos\delta}$$

$$= \frac{\cos\delta'}{\sqrt{n^2-\sin^2\delta'}}\cdot\frac{\sqrt{n^2-\sin^2\delta}}{\cos\delta}.$$

and $$\sin\delta = \frac{b\sin\delta'}{\sqrt{R^2+b^2-2Rb\cos\delta'}}$$

$$\cos\delta' = \frac{R-b\cos\delta'}{\sqrt{R^2+b^2-2\,Rb\cos\delta'}}$$

† The angles δ and δ' being less than 38°, the co-efficient of crystalline reflection is sensibly constant in these mits, and we can give them the same value, 4½ per cent., as at zero incidence.

We thus obtain the following table :—

CALCULATION OF THE CO-EFFICIENT k OF THE PRECEDING MIRRORS.

Radius of each parallel in metres.	Thickness of glass traversed in millimetres.		Co-efficient of transmission in the glass.		Co-efficient of transmission at the surfaces of passage and on the silver, 1st and 2nd.	Resulting co-efficient k.	
	1st.	2nd.	1st.	2nd.		1st.	2nd.
0·00	15	10	0·955	0·970	0·955 × 0·90 × 0·955 = 0·821.	0·77	0·79
0·10	16	12	0·952	0·964	"	0·77	0·78
0·20	19	18	0·943	0·946	"	0·76	0·77
0·30	24	28	0·938	0·916	"	0·76	0·74
0·40	32	43	0·904	0·871	"	0·73	0·71
0·45	37	53	0·889	0.841	"	0·72	0·68

The diagram, (Fig. 20) represents in the form of curves the co-efficients, k, u, w, of the two mirrors.

We see that the coefficient, u, increases while k diminishes, so that the product, ku, is nearly constant.

It has the same value 0·73 at the edges for the two projectors, and the values 0·77 at the centre of the first and 0·79 at the centre of the second.

We can take as means 0·74 and 0·75, a value which is from 5 to 10 per cent. smaller than that of mirrors in thin silvered glass.

The divergence is defined by the ordinary ratio. I have reproduced in a comparative chart the divergences of two parabolic projectors of the same focal lengths: they diminish much more rapidly from the centre to the edge of the mirror than for a Mangin reflector of the same focal length.

5°. *Catadioptric rings.*—The prisms, C D, for total reflection invented by Fresnel to complete the annular lenses, (Fig. 22) have very sensibly $\frac{d\omega}{d\omega'}$, = 1 (within about 5 per cent.). We can therefore apply to them all that has been said about parabolic mirrors making k=0·70 to 0·80 according to the position and dimensions of the ring.

Graphic Integration.—The preceding calculations show how we can obtain in the form of curves the factors, k and u as functions of r. If we know the value of i as a function of r, *i.e.*, the brightness of the source in the direction of the ray meeting the parallel, r, we can calculate the values of the product kui itself and represent them by a curve in rectangular co-ordinates as a function not of r but of r^2. The expression for the power admitting of being written in the form

$$P = \pi \int_{r_0}^{r_1} (kui)\, d(r^2),$$

we see that P will be obtained in measuring the area between the curve (kui) and the axis of r^2 which permits of completing the calculation in each special case.

But I will remark that it is not generally of much use to make this rigorous integration, for the way in which i varies according to the direction of the ray always presents an amount of uncertainty, because we are never sure in practice that the crater of an arc is uniformly brilliant: on the other hand, the product, ku, varies so little with r, at least in mirrors, that we can, without much error, attribute to it simply a mean value as will be seen below.

IV.

GENERAL CONSEQUENCES OF THE THEORY.

Practical Coefficient of Utilisation.—We have seen that in every case the luminous power is of the nature of the product of a surface by an intrinsic brightness.

A mirror or an aplanatic lens behaves at a great distance exactly as a plane incandescent surface, having for intrinsic brightness at each point the same brightness as the source at the corresponding point multiplied by the coefficients of transmission, k, and of optical effect, u.

If we are working with a source whose brightness is well defined, such as an arc with continuous current, of which the crater alone plays an appreciable part, we can practically replace the exact formula (5) by the following empirical formula :—

$$P = \lambda\, S . i_{\text{mean}} \qquad \qquad (21)$$

S being the projection of the utilized surface of the projector on a plane perpendicular to the optic axis, and λ a ***numerical*** coefficient which we may call the coefficient of utilization ; $\lambda =$ mean (ku).

It is not rational to express this power as a function of the luminous intensity of the source, as is done by most makers, instead of preserving the form to which the preceding calculation leads. M. l'Inspecteur général Bourdelles was the first to point out the error as far as concerns lenses,* and has substituted for the old coefficients of the instruments, coefficients which permit of referring the power directly to the mean brightness of the source, without any hypothesis as to its form.† The same notation should be used for projectors, and it is formula (21) which should be applied exclusively to them. The coefficient, λ, which represents the reduction of brightness due to the optic effect and to the absorption of the instrument will be obtained by the preceding calculation, or empirically by photometric measure as well as the useful surface S.

Rôle of the different elements of a projector and of its lamp.—The formula no longer containing the focal length, we see that the luminous power is itself independent of it, at least for parabolic projectors‡ in the case of the theoretically free crater (Fig. 2). This question, being very

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* M. Bourdelles has given for the intensity of light-houses the two following formulæ, calling K and K' two coefficients, d the diameter of the source, ω the horizontal opening of the lens, n a coefficient of reduction, f the focal length,

fixed light . . . $P = Ki \dfrac{\pi d}{n} . f$

flash „ . . . $P = K'i \dfrac{2\omega}{n.1800} . f^2$.

He has determined the values of the coefficients K for the different instruments of light-houses and the mean intrinsic brightness of the lamps employed in these instruments, and has deduced important practical conclusions. For further details see his able Memoir, "The Luminous Power of Light-house Apparatus" at the Maritime Congress of 1893.

† It is only at the price of an hypothesis of this kind that Mangin, after having indicated the formula for lenses given above and analogous to (21), has been able to introduce directly the notion of power amplifier which actually passes current, and which establishes a relation between two quantities theoretically independent of each other.

‡ For other instruments the coefficients, k and u are a little modified but not sufficiently to change the conclusion for practical purposes : for Mangin's mirror we have already seen that the product, ku, remains sensibly constant when the focal length is changed.

important, will be treated further on with the necessary details. The coefficient, λ, depends, as we have seen, on the type of projector employed. Theoretically, it is maximum for the parabolic projector; practically, it depends, besides, on its precision and execution, *i.e.*, on the residual aberration and on the chromatic aberration. As for the useful surface, it depends not only on the diameter of the apparatus, but also in a very large measure on the form and disposition of the source of light. I will pass rapidly in review the different elements.

1°. *The surface of the mirror* is evidently one of the most important elements. If there was no occultation the luminous power would be proportional to this surface: there is a great object in increasing the opening of projectors, but it appears difficult to much exceed the actual present dimensions.

2°. *The aberrations* have the effect of substituting for a single focus, F, (Fig. 18) several foci distributed in the focal plane between the two extreme points, $F' F''$, the only chromatic foci to consider are, moreover, those corresponding to the true lighting colours (from red to green, inclusive), the others being very rapidly absorbed by the atmosphere. To completely utilize the instrument, a source of light is necessary which unites all these rays, *e.g.*, an arc lamp with horizontal carbons whose crater is placed in the plane where the length, $F' F''$, is minimum, and has at any rate the dimension $F' F''$. The intensity of current necessary to realize in an instrument the maximum power is regulated only by this condition. So long as the crater is smaller than $F' F''$, the power practically increases with the intensity of the current; once it is greater, there is no further advantage in increasing it, always admitting that the arc remains geometrically and physically similar; the augmentation of the intensity only makes itself more felt on the amplitude of the pencil, a fact which appears to be verified by experience, and is consequently only of use when we wish to increase the illuminated angle without reducing the intensity of illumination. It is needless to say that this spreading out of the pencil demands a great outlay of electric energy.

We need not trouble in practice about the effect of chromatic aberration in lenses and refracting mirrors. In fact we may remark at once that, in consequence of the superposition of elementary pencils, the coloured rays combine to form white light, and that the resulting iridescence of dispersion is only sensible on the edges themselves of the pencils: it is chiefly marked in the pencil of lenticular projectors, in refracting mirrors it is much less sensible. Experience shows that this iridescence is without practical inconvenience, and that, besides, it is rapidly extinguished by atmospheric absorption when the distance is increased. At a distance of 400 or 500 metres from the projector, where the resultant blending of the superposition of the elementary pencil is effected, the section of the pencil no longer presents any iridescence appreciable to the eye.

3°. *The arrangement of the crayons of the lamp* makes itself felt in the way in which it frees the crater with regard to the surface. Lamps with horizontal carbons (Schuckert, Breguet) do not illuminate the centre of the mirror, and, on the contrary, give all their action to the edges. Lamps with inclined carbons (Sautter-Harlé) tend to produce the opposite effect. The difference between these two modes of lighting can be appreciated in a certain measure from the curves of luminous intensity of Figs. 17 and 23 which I have reproduced here from the measures of M. J. Rey, and from a study of M. Nerz. But the true distinction is the following:—

The only useful points of the optic surface being those whose focal rays freely meet the crater, all that part of this surface situated at the interior of a cone which I call the *cone of occultation* (not to be confounded with the cone of shadow), having for the vertex the focus, F, (Fig. 24), and tangent to the negative crayon, is practically useless.

Now, this cone of ocultation, which has an opening of 30° to 40° masks the lower edge of the mirror when an oblique lamp (Fig. 17) is employed, or the centre, when a horizontal lamp (Fig. 23) is used (the cone is covered with hachures): consequently the employment of lamps with oblique crayons appears to me preferable for instruments with long foci, while a lamp with horizontal carbons ought to be just as good for a large instrument with a short focus. Conversely, it is inconvenient or profitable to shorten the focus of an instrument according as it is illuminated by an oblique or horizontal lamp: Fig. 23 shows, for example, that a mirror, A_2 B, with a long focus would have a lost surface eight times greater than the mirror A_1 B_1.

In both types of lamp care should be taken to reduce the cone of occultation to its minimum value by employing a negative crayon of as small diameter as possible, and of an arc as long as possible, while taking into consideration the other practical conditions.*

4°. *The intrinsic brightness of the positive crater* plays an extremely important part since it figures as a factor in the expression for the power. We cannot wish to employ a source of light of greater brilliancy than the arc with continuous current, but we ought to avoid all that can lower that brightness below its maximum. I have recently shown that it can be reduced enormously by employing too good conductors in the composition of the pencils. We ought therefore to employ in the carbons only just enough of foreign salts to obtain a good stability. We ought also to augment as much as possible, the density of the current to obtain a crater well saturated presenting a mean brightness nearly the maximum.

5°. *The focal length* has no direct influence on the power in the case of a source without occultation. It has an influence on the amplitude of the pencil and on the *total light* which it contains, both varying inversely with the focal length according to a law which depends on the form of the projector and of the curve of intensity of the lamp: but the arguments which precede have shown that there is no need to seek to establish a relation between this quantity and the luminous power.

It is so much the more necessary to insist on this point because there are few on which so much discussion has been raised, and because several recent works containing reasoning which does not stand a serious analysis have, nevertheless, thanks to the elsewhere justified authority of the authors, been able to create prejudices on the question in many minds.

I repeat, therefore, what I have said above: *The focal length, not entering into formula (21), cannot have any theoretical influence on the luminous power in the case of a perfectly free crater.* This agrees with the fact that the divergence increases at the same time as the quantity of light received. As Colonel Mangin had foreseen, a projector of given surface fed by non-occulted source, such as an oblique lamp, gives a pencil more or less open, and lights, consequently, a larger or smaller space according as we reduce or augment its focal length without modifying the intensity in the axis of the pencil. When we employ a lamp producing occultation, this conclusion can be modified in the sense that has been indicated in the preceding paragraph, but *that proceeds entirely from the form of the cone of occultation* and not at all from the optic properties of the projector. We can therefore state the following general principle: *The power can be increased by the reduction of the focal length only when this proceeding permits the reduction of the occulted surface.*

But it must not be forgotten that this reduction of focus has the inevitable result of increasing the amplitude of the pencil. Short foci are therefore to be recommended only in the case where a very open pencil is desired: we see that they then present an inconvenience so serious that it is necessary to point it out.

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* The reduction of the diameter of the negative is limited by its heating and by its rapid using up: the gap by the instability of the arc and the augmentation of the corresponding loss of energy. The values adopted for these two elements by makers are those that practice has indicated as best reconciling these opposing conditions. The oblique lamps of the Sautter-Harlé type work, for example, to 65 amperes under a gap of 10 to 12 millimetres, corresponding to 52 or 53 volts, and to 100 amperes under a gap of 15 millimetres and 56 to 57 volts as limits: in the latter case the diameters of the crayons (Fig. 24) are 25 and 18 millimetres. These voltages higher than the industrial ones are necessary to make use of every possible part of a projector. There is, besides, a little sleight-of-hand for oblique which allows of obtaining the best shape for the positive. With a horizontal lamp the gap cannot be quite as great, and that is an inconvenience because a horizontal lamp always presents less stability than a vertical one: so makers indicate a lower voltage (48 volts for Breguet lamps).

6°. *The form of the instrument* for equal apparent surfaces affects the composition of the pencil, that is to say, the way in which the intensity of illumination varies from the centre to the edge. The rigorous calculation of this distribution is easy in the case of a non-occulted source of light, but it suffices here to see how the minimum divergence of the infinitely small pencils previously considered (Fig. 7), varies from the centre to the edge of the projector. This variation is represented by the values of the coefficient, w, in the tables and by the corresponding curves of Fig. 21. From what was said on page 28, the total section illuminated by the pencil is proportional to the maximum divergence, and the diameter of the central spot (where the illumination is maximum) to the minimum divergence. We see that *for equal focal lengths* the three types of principal projectors (lenticular, parabolic, aud Mangin's) have the same maximum divergence and consequently illuminate the same total surface, but that the central spot is greater, and consequently the pencil non-homogeneous, in refracting mirrors than in parabolic ones.

Fiaally, Fig. 21 shows clearly how the ratio between the minimum and maximum divergence decreases rapidly with the focal length. In the parabolic mirror of 34 c.m. this ratio is not more than 0·28. We have therefore a pencil open and heterogeneous whose central spot is less extended than that of a mirror of 1·015 m. Fig. 21 represents, comparatively on a common scale, the sections of pencils corresponding to the five cases considered and gives a clear view of the composition of the pencils in each of them. Fig. 25 shows in a still clearer way the effect of the focal length of a parabolic projector of 0·900 m. on the composition of its pencil.

It is therefore necessary to guard against too great a reduction of the focal length, *particularly in parabolic mirrors.*

For these it does not seem reasonable in ordinary circumstances to reduce the focal length below *the value which makes the diameter of the central spot maximum.* This value, as we have seen above, is $\frac{6}{10}$ of the opening, (useful diameter), *i.e.*, 0·54 m. for a mirror of 0·900 m. of opening. Besides, the heating of the projector is a no less important consideration which we must take into account, and which does not allow the focus to be brought indefinitely near the reflecting surface.

Application of the formulæ to the prediction of experimental results.—Some attacks having been made on photometry * *àpropos* of the figures of power given for the projectors at the Chicago Exhibition, I cannot choose a better example than that of these projectors to show that a simple knowledge of formula (13) would have saved all discussion as to their order of merit.

Let us consider, in fact, a parabolic reflector of 1·500 m. (the useful diameters of the great projectors of Mangin and Schuckert) having a plane surface of 1,767,000 mm.^{2}; take for the coefficient of transmission of silvered glass the mean figure 0·85, and for the intrinsic brightness of the positive crater the figure 160 candles per mm.^{2} which I have recently indicated (provisionally).† The diameter of the surface rendered useless by the cone of occultation is very nearly—

$$D = 2f \sin 17^\circ 30' = 0{\cdot}39 \text{ m}$$

the focal length being 0·65. The surface occulted is thus $\frac{\pi D^2}{4} = 0{\cdot}1194 \text{ m}^2$. We have, therefore, by formula (21)—

$$P = 0{\cdot}85\,(1767000 - 119400) \times 160 = 225{,}000{,}000 \text{ candle power.}$$

Admitting, even, that the mean brightness of the crater may be only 130 candles, we still find 181 millions candle power, that is the same intensity as furnished directly by a crater of arc having a superficies of $181 : 130 = 1{\cdot}40 \text{ m}^2$. The figure 190 million candle power announced by the maker, which appeared so extraordinary, is therefore very probable, and a parabolic mirror ought to realize it, if properly made, which, of course, remains to be demonstrated by direct measures.

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* *Industrie électrique* of 10th November, p. 499.

† "On the arc with continuous current with regard to its employment as a standard of light." Congress of Chicago, 1893. This figure of 160 candle power greatly surpasses all those of previous observers.

Comparison of divers actual types of projectors.—The best projector is in general that which, with equal opening and the same lamp, gives the greatest power and the most homogeneous pencil. It is therefore the one which gives—

1°. The best coefficient λ, that is to say, the best coefficients k and u.

2°. The smallest aberrations.

3°. The most homogeneous pencil.

Theoretically, the parabolic reflector is superior to others in the first two points for it is exempt from spherical and chromatic aberrations, it has the smallest loss by absorption and reflection, and its coefficient, $u = 1$; finally, we can give it any focal length so as to obtain very divergent pencils (but these are wanting in homogeneity as we have seen).

On the contrary, the refracting mirror presents on account of its thickness a smaller coefficient, λ, (0·73 to 0·75 instead of 0·80 to 0·85 for the parabolic) whilst u may be a little greater than 1; and the value of its focal length is restricted. Yet the useful focal lengths of late years are not much greater than those of parabolic mirrors constructed in trade: they are in every case very little different from that which I have determined above as the most advantageous. Besides we have seen that the focal length does not reduce the power with an oblique lamp and consequently we cannot object to mirrors with long foci for use with such lamps. Medium foci are also the most advantageous when we wish, as generally occurs in practice, a pencil of small opening and very homogeneous. From the point of view of homogeneity, the refracting reflector is superior to the parabolic mirror, as we have seen above, for with equal focal length it gives a larger central spot.

Practically, the value of a projector depends chiefly on the perfection with which it is made, and from this point of view the Mangin mirrors have hitherto been the most remarkable, thanks to the employment of circular meridians which can be better made than any others, so that the supplementary aberrations due to errors of construction do not exceed the order of magnitude of theoretical aberrations,* that is to say, a fraction of a millimetre for the largest mirrors. Parabolic mirrors are of a much more difficult shape, and can practically lose, by their aberrations of construction, the advantage due to their form.† It is therefore on the skill of the makers that the superiority or inferiority of one type of mirror with regard to the other depends, rather than on their theoretical properties.

In the same way, in practice, the questions of the thickness and of the chromatic aberration have very different effects from that given by theory. The chromatism disappears completely at a certain distance, and the thickness is very advantageous on account of solidity. One of the objections which can actually be made to parabolic reflectors is the fragility which results from their small thickness and which thus discounts the advantages from the point of view of absorption. The thickness can only be increased by completely changing the method of construction of these instruments.

I believe, therefore, that the choice of one of the two types of mirrors is a question of kind, and that the definite superiority of one or the other is not so well established as is sometimes stated.

As for lenses in echelon, they are very inferior to mirrors because their residual and chromatic aberrations are always considerable, their coefficients, k, and u rather feeble, and their framing delicate, rendering them liable to displacement.

This is why they have almost been abandoned for the projection of light, and their use is confined to light-houses. Here also, however, thanks to the admirable system of *flash lights* recently invented by M. Bourdelles, the mirrors replaced at the commencement of the century by Fresnel's lenses, and considerably improved since then, may be able, perhaps, to again play a part which the old instruments of rotation denied them.

* See above, p. 40, note.

† This fact has been shown by the remarkable photographic researches of M. Tchikolev.

The conditions of employment, and, consequently, the considerations which can guide in the choice of a reflector in view of this application, are absolutely different from those that I have just developed, and cannot find a place here.

RÉSUMÉ AND CONCLUSIONS.

I hope I have shown in this paper that the photometry of projectors can be treated in a rigorously scientific and yet simple manner. With this object, after having established that a projector should be defined, from a photometric point of view, by its *luminous power* on the axis at a great distance and by the *constitution of its pencil*, and having indicated a general theorem applicable to the determination of these, I have shown how the calculation of the first is reduced to that of *the apparent brightness at each point of the surface of emission*, and the study of the second to that of the *elementary divergences*. The results obtained for each kind of projector enable us to represent, with an approximation practically always sufficient, the value of the luminous power by a single empirical formula, very simple and applicable to all the types, and also to completely analyse the influence of the various optical elements and those of the lamp. The result in particular of that analysis is that the rôle of the long focus is notably different from that which is generally assigned to it.

Finally, I have established between the principal types of mercantile projectors a comparison which I have endeavoured to keep quite impartial. I will not pronounce in favour of one projector more than another, for (and I here stop to insist once more on this point) *the adoption of a mirror of short or long focus and of an oblique or horizontal lamp must essentially depend on the circumstances of employment and on the kind of service which one proposes to get from the instrument*, and should, in each separate case, result from a well reasoned out comparison for which the conclusions developed above have only the aim of furnishing the elements. As far as concerns projectors for use in the army or navy, the officers of the different arms alone are competent to make the comparison. I will be happy if these notes, in spite of their brevity, remove the misconceptions and prejudices which have found their expression in recent pamphlets and discussions, afford useful hints to those who have to construct or employ projectors, and so contribute in a modest way to that industry which is so essentially French since it is our fellow countrymen, engineers, constructors, officers on land and sea who have initiated it, made the greatest progress, and propagated it in almost the whole world.

Translated by
J. ECCLES.

ANNEXE.

ON THE ILLUMINATION PRODUCED BY A PARABOLIC MIRROR BY MEANS OF A SPHERICAL AND OF A PLANE SOURCE OF LIGHT.

BY JEAN REY.

1°.—*The Source being spherical.*—Let AOB, (Fig. 26) be a parabolic mirror having its focus at F. Let M be a point on the mirror, M P the reflected ray coming from the focus, ρ the radius of a spherical luminous source placed in the focus, δ the angle made by the ray with the axis, and a the angle made by the normal either with the radius vector or with the axis. It is supposed that the luminous source follows Lambert's law, which need not be here enunciated. By virtue of this law, a luminous sphere appears to the eye as a plane disc, and a luminous spherical source is equivalent to a plane disc of the same diameter and of uniform brilliancy. The luminous source under consideration has then an intensity which is constant in all directions. This intensity is equal to the surface of the plane disc multiplied by the intrinsic brilliancy—that is to say, by the brilliancy per unit of surface.

In order to facilitate these calculations it is necessary to make an hypothesis which is perfectly legitimate, *viz.*, that the size of the source is negligible in comparison with that of the mirror, or, in other words, the angles made by the rays leaving the edges of the source with the normal to a point on the mirror are to be considered as equivalent and equal to the angle made by the luminous ray coming from the centre of the source with the normal to the same point.

To evaluate the illumination produced in the central zone of the beam it is necessary to calculate separately the illumination produced by each part of the mirror. If x and y be, respectively, the abscissa and ordinate of the point, M, the equation to the parabola when its vertex is taken as origin is $y^2=4fx$, f, being the focal length. Let F M $=l$ (the radius vector), and let us consider the surface of the elementary zone containing the point, M. If $m\ m'$ be an element, (ds), of the parabola, then the surface of this zone is $2\pi y\ ds$. It receives light of which the intensity is I, where I is the constant intensity of the spherical source.

To obtain the flux of light falling upon the zone under consideration we must multiply the intensity which it receives in the direction of the radius vector by the solid angle of this zone. This solid angle is the projection of $m\ m'$, in the direction of the luminous ray, divided by the square of the distance. The value of this angle is therefore $\frac{2\pi y ds \cos a}{l^2}$. The flux through this solid angle is then $\frac{2\pi y ds \cos a}{l^2} \times I$.

We must now find the illumination produced by this flux. Each of the points, M, reflects a luminous cone having as its base the image of the luminous sphere—that is to say, a plane disc of the same diameter has such a sphere. Each reflected cone has therefore a circular base. At a distance, D, the radius of the section of the reflected cone is $\rho\frac{D}{l}$ and the sectional area of the conical elementary beam is $\pi\frac{\rho^2 D^2}{l^2}$. Since now the elementary reflected conical beams proceeding from all points of the zone cross in space at even a moderate distance and form themselves soon into a single beam, we may consider that the zone under consideration produces at distance, D, a circle of illumination, which illumination is the flux received by the zone divided by the sectional area of an elementary beam calculated as above. This illumination is therefore—

$$\frac{2\pi y ds \cos a \times I}{l^2} \div \pi\frac{\rho^2 D^2}{l^2} = \frac{2I}{\rho^2 D^2} y ds \cos a.$$

As we approach the edge of the mirror, the luminous cones reflected from the elementary zones become more acute. The elementary zone at the very edge of the mirror produces the smallest circle of illumination, hh'. It is precisely this circle which governs the dimensions of the central zone, $kk'=hh'$. As all the other zones produce larger circles of illumination which superimpose themselves on the circle, hh', it is only this latter circle which receives light from all the zones of the mirror. On the other hand, the centre of the mirror produces a large circle of illumination, the edge of which, nn', receives light only from the centre of the mirror itself. To obtain, then, the illumination of the central zone of the beam, it becomes necessary to find the sum of the illuminations produced by all the elementary zones of the mirror—that is, it is necessary to integrate the expression $\frac{2I}{\rho^2 D^2} y ds \cos a$ and between the limits $y=0$ and $y=r$ (the semi-diameter of the aperture of the mirror). It is clear that $d\,s \cos a=d\,y$, and the illumination of the central zone becomes—

$$\frac{2I}{\rho^2 D^2}\int_0^r y\,dy = \frac{2I}{\rho^2 D^2} \times \frac{r^2}{2} = \frac{I}{\rho^2 D^2}r^2.$$

If i be the intrinsic brilliancy of the source, then $I=\pi\rho^2 i$, and if we call the illumination of the central zone, E,

$$E = \frac{\pi\rho^2 i}{\rho^2 D^2} \times r^2 = \frac{\pi r^2 i}{D^2} \text{ or } \frac{i\,S}{D^2},$$

where S is the area of the aperture of the mirror. We may say, then, that the optical power of the mirror, $E D^2$, is i S, or the intrinsic brilliancy of the source multiplied by the area of the aperture of the mirror. For a spherical source of given intrinsic brilliancy the optical power of a parabolic mirror is therefore constant, and is independent of the focal length.

2°.—*The Source being plane.*—If the luminous source be plane, and follow the law of Lambert the intensity in a direction making an angle δ with the axis is $I_0 \cos \delta$, where I_0 is the intensity normal to the plane source. This is exactly what happens with electric arc lamps, and, be it well understood, there exists no such thing in reality as a spherical luminous source.

Let us now consider a point of our parabolic mirror illuminated by a plane source. The point is illuminated by a cone of rays having the plane circular source as its base. The right section of this cone through the focus is, however, no longer a circle, but an ellipse, the major axis of which is 2ρ, and the minor axis of which is $2 \rho \cos \delta$. The point, M, reflects a luminous cone, the right section of which in space is a similar ellipse. Now, the dimensions of the mirror are small compared with those of the circle illuminated, at any considerable distance, and we may therefore suppose, as far as the illumination of such a circle be concerned, all points of any one zone of the mirror to be concentrated at the centre of the mirror. These points will consequently produce a series of elliptical reflected cones, the right sections of which have the same dimensions, but which are situated with regard to one another as indicated in Fig. 27. We see, then, that all the elliptical cones of which the whole beam of the zone is built up have one common part, which is a right cone having for diameter of base the minor axis of these ellipses.

Let us now calculate the illumination produced by the circular zone, mm', mm'.

Its solid angle is, as we have seen, $\dfrac{2 \pi y \; ds \; \cos a}{l^2}$.

The flux received by this zone is now, however

$$\frac{2 \pi y \; ds \; \cos a}{l^2} \times I_0 \cos \delta = \frac{2 \pi y \; dy \; \cos \delta}{l^2} I_0.$$

This flux, after reflection, falls upon an ellipse whose axes, at a distance, D, are $2 \rho \dfrac{D}{l}$ and $2 \rho \cos \delta \dfrac{D}{l}$. The area of such ellipse is $\dfrac{\pi \rho^2 D^2 \cos \delta}{l^2}$.

The illumination is therefore

$$\frac{2 \pi y \; dy \cos \delta \; I_0}{l^2} \div \frac{\pi \rho^2 D^2 \cos \delta}{l^2} = \frac{2 \; I_0 \; y \; dy}{\rho^2 D^2}.$$

This expression is exactly the same as that obtained in the case of the spherical source, and if we again integrate between the limits $y = 0$ and $y = r$, we obtain the value,

$$E = \frac{I_0}{\rho^2 D^2} r^2 = \frac{\pi r^2 i}{D}, \text{ or } \frac{i S}{D^2}.$$

Thus, in this case also, the luminous power of the mirror, $E D^2$, is the product of the area of its aperture multiplied by the intrinsic brilliancy of the source. In fact, whatever be the form of the source, the focal length will not enter into the expression for the luminous power. For an ordinary lens, or for a Mangin mirror, it would be necessary to ascertain for each successive zone the exact divergence of the reflected cone; the formulæ become very complicated, and integration becomes impossible.

The above is rigorously exact as long as the size of the source may be considered negligible as compared with the size of the mirror. If the brilliancy of the source be not uniform all over, the value of the intrinsic brilliancy in the ultimate formula would be a mean which it would be necessary to evaluate by the aid of the integral calculus; but this would not change the basis of the reasoning.

G. I. C. P. O.—No. 100 S.G.—15-8-03—C. M. W.

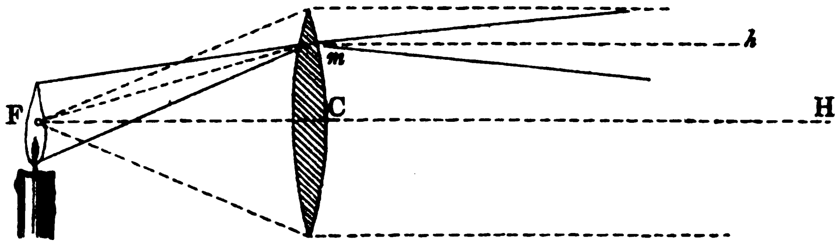


Fig. 1

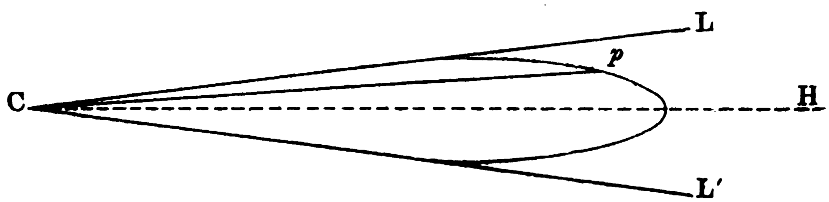


Fig. 2

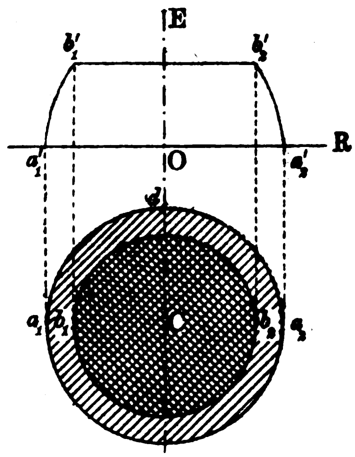


Fig. 3. Section of a pencil of an electric projector at a great distance and curve representing the variation of intensity of illumination along a diameter of this section. b_1, b_2, central spot uniformly illuminated.

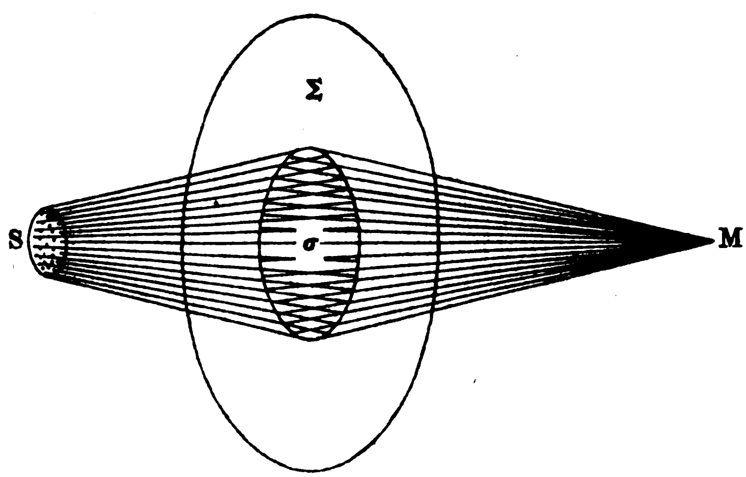


Fig. 4.

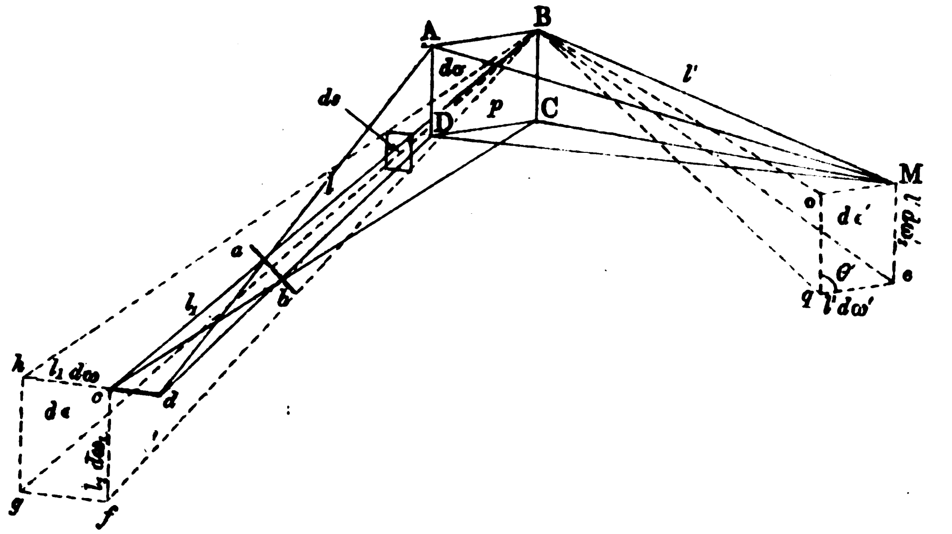


Fig. 5.

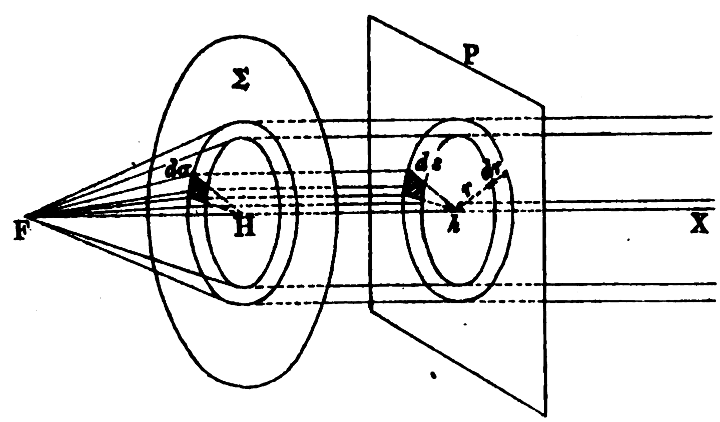


Fig. 6.

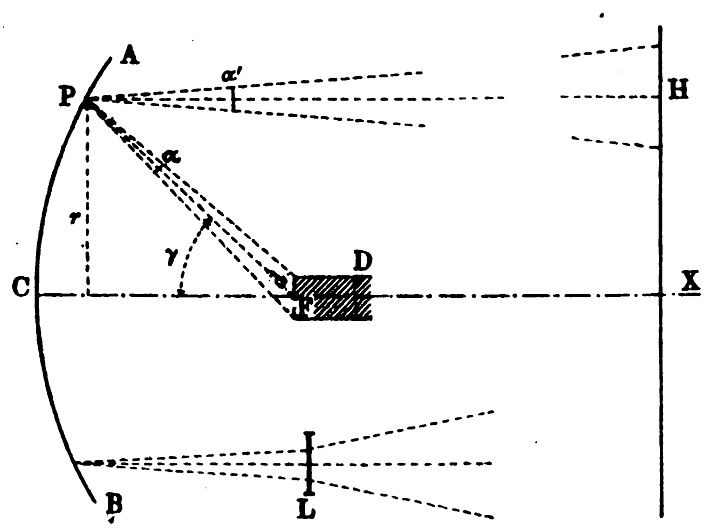


Fig. 7. Mirror lighted by a theoretical arc reduced to a circular crater *d*.

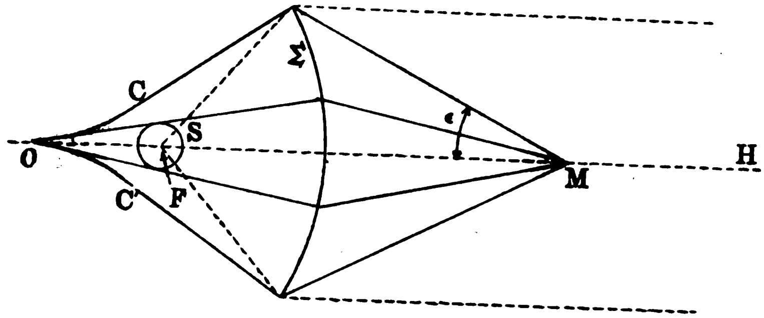


Fig. 8.

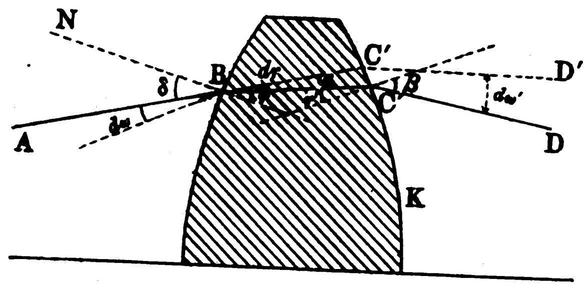


Fig. 9.

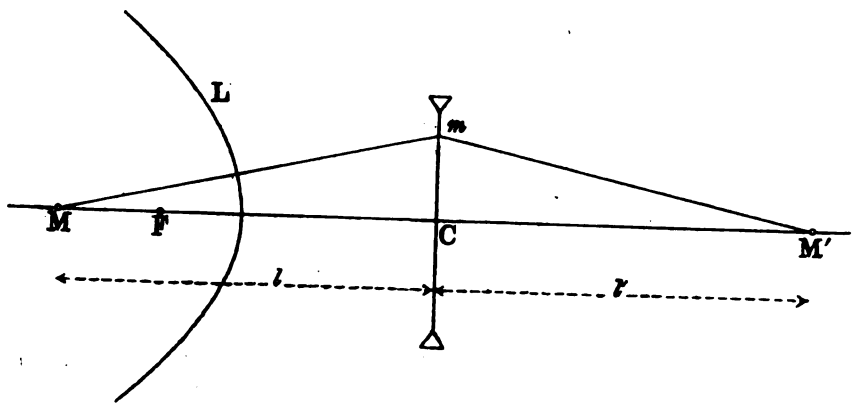


Fig. 10. Theoretical Lens.

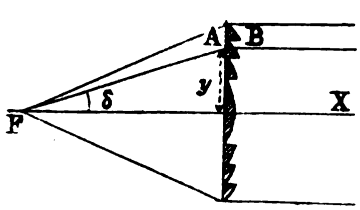


Fig. 11

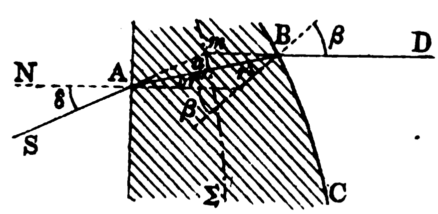


Fig. 12.

Figs: 11 & 12. Lenses in echelon.

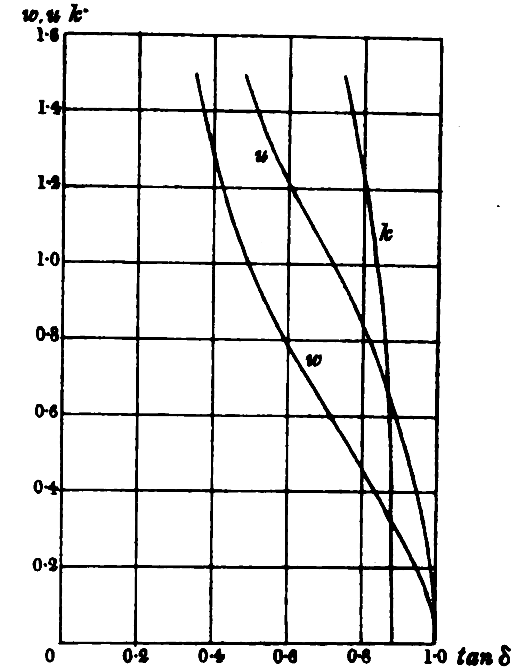


Fig. 13. Values of the coefficients u, k, w, as a function of tan δ for a lens in echelon of 1·00 metre.

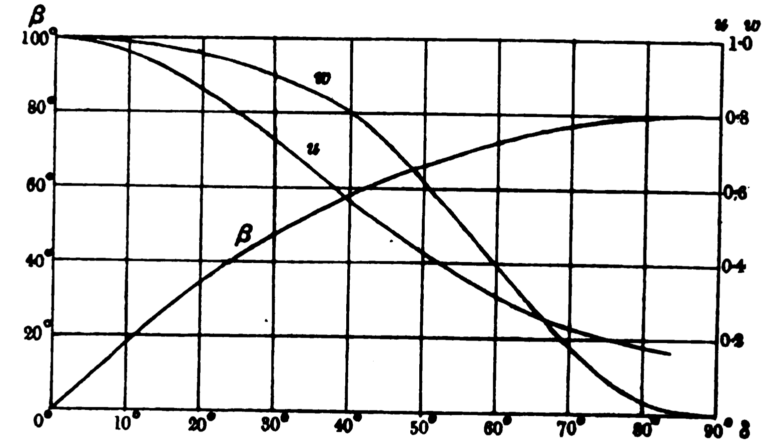


Fig. 14. Values of angles of emergence β, and of coefficients u and w as a function of the angle of incidence δ for a lens in echelon.

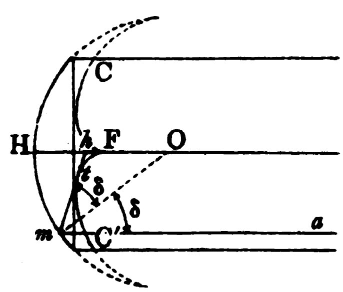


Fig. 15. Spherical Reflector.

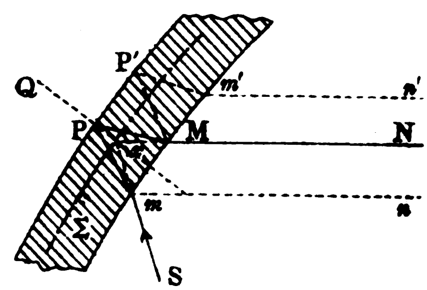


Fig. 16. Section of a mirror of thin silvered glass.
argente mince.

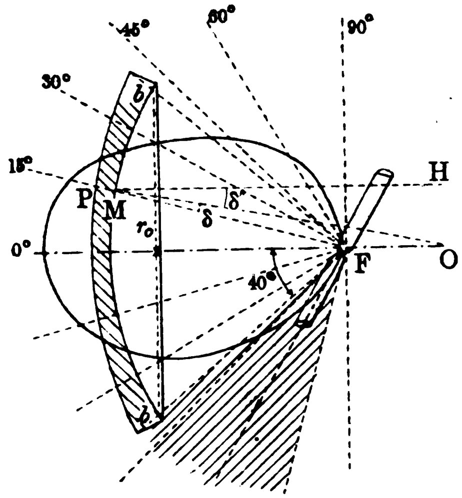


Fig. 17.

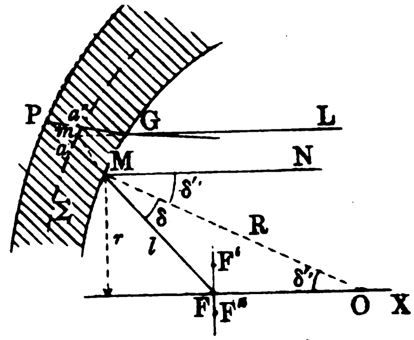


Fig. 18.

Figs. 17 & 18. Refracting reflector of Col. Mangin and lamp with inclined carbons.

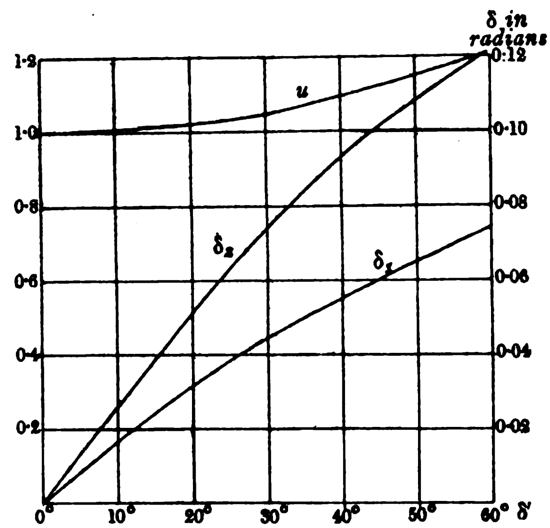


Fig. 19. Values of the angle of incidence δ and of the coefficient of optical effect *u* as a function of the angle of emergence δ' for the two Mangin projectors of 0·900.

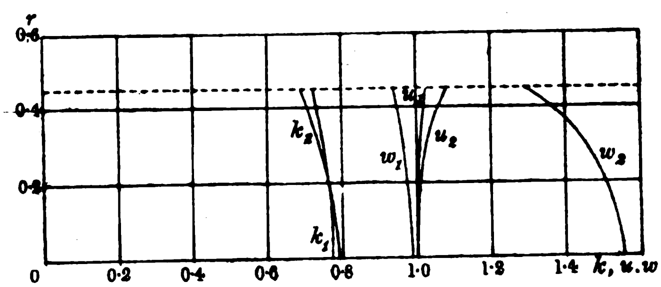


Fig. 20. Values of the coefficients *k*, *u*, *w* for the two Mangin mirrors of 0·900, the first having a focal length of 1·015 and the second of 0·645.

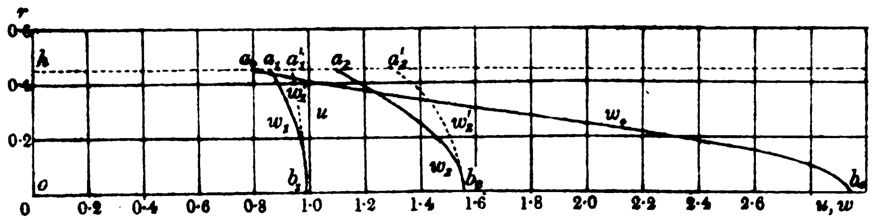


Fig. 21. Comparison between the divergences of refracting mirrors and parabolic ones w_0, w_1, w_2, coefficients of paraboloids of 1·015, 0·645 and 0·34 focus w_1, w_2, coefficients of Mangin mirrors of same focus. The lengths ob_0, ob_1, ob_2, are proportional to the total diameters of the illuminated circles and the lengths ha_0, ha_1, ha_1, ha_2, ha_2, to the diameters of central spots.

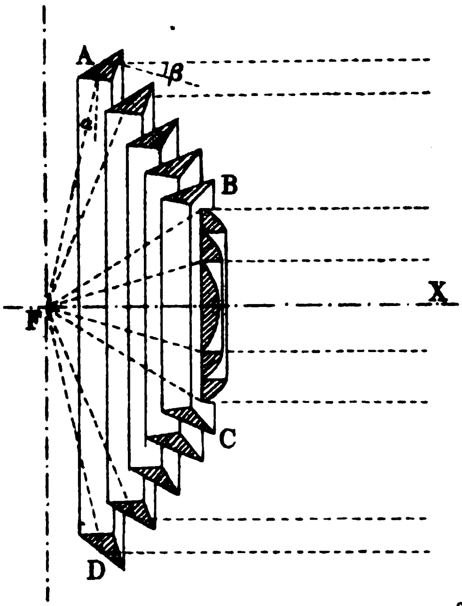


Fig. 22. Complete lenticular projector.

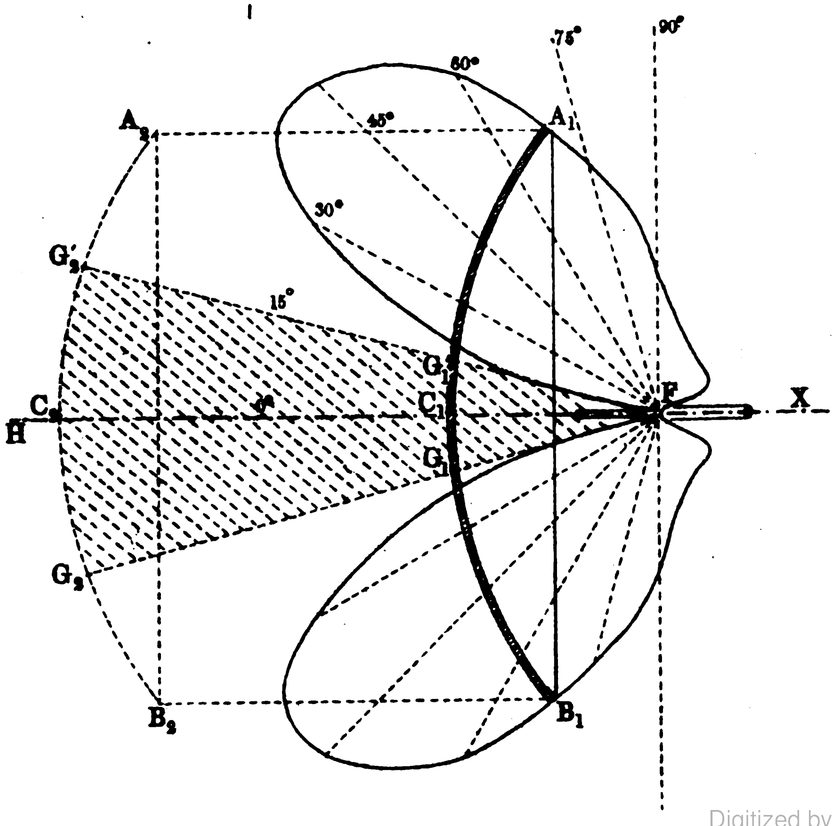


Fig. 23. Parabolic Reflector and Horizontal Lamp.

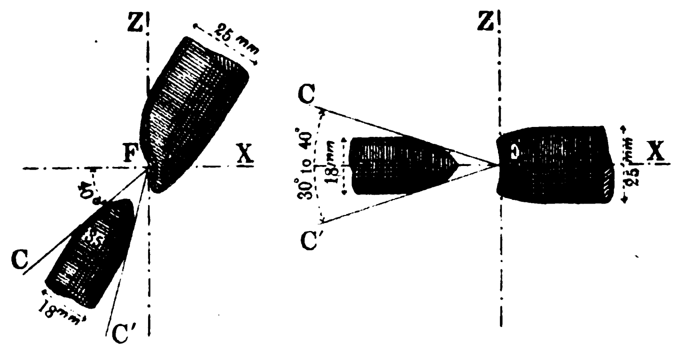


Fig. 24. Determination of the cone of occultation and form of the crater in the two types of lamps.

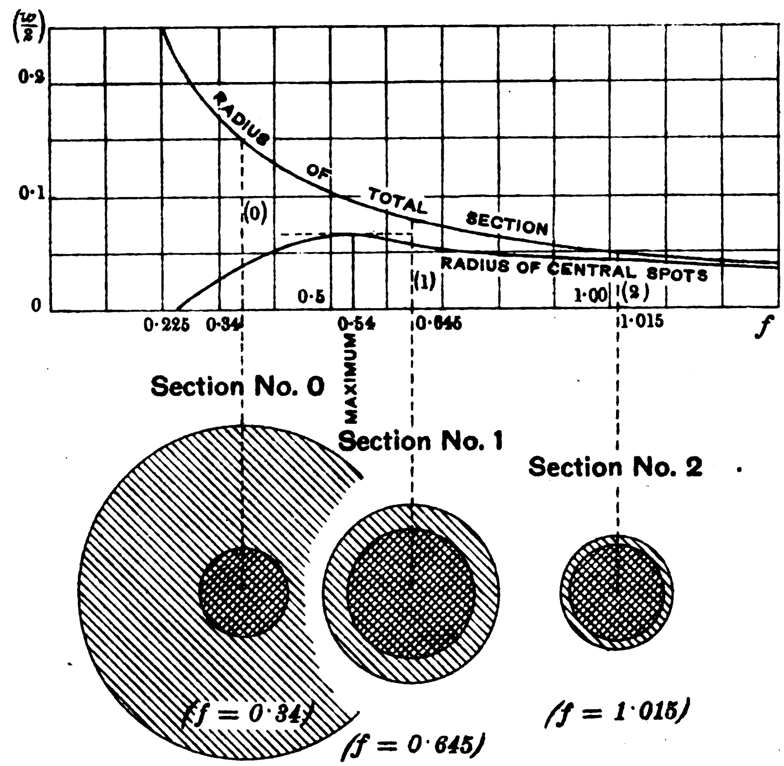


Fig. 25. Comparison of the spots of illumination obtained with a parabolic projector of 0·900 m. opening for different focal lengths. Variation of the radius of the section and of the central spot as a function of f

Fig. 26

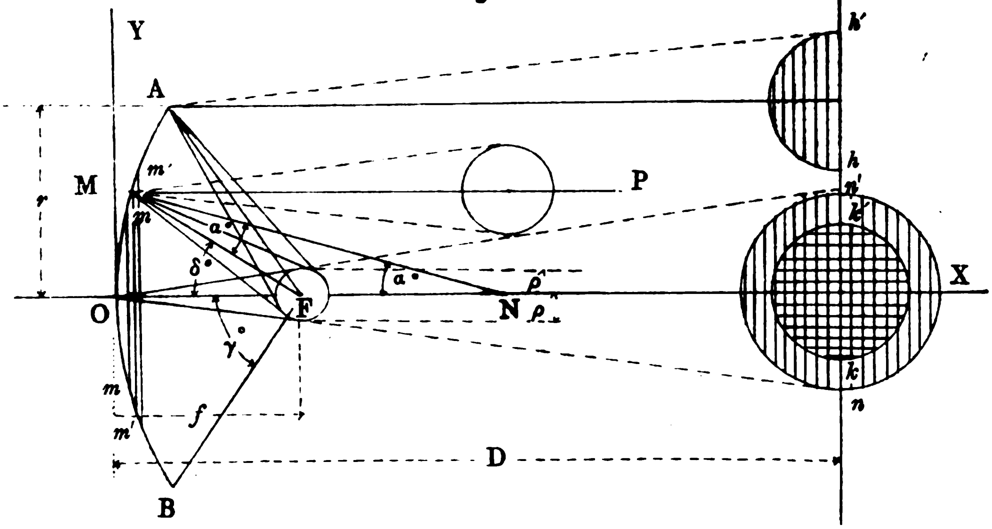
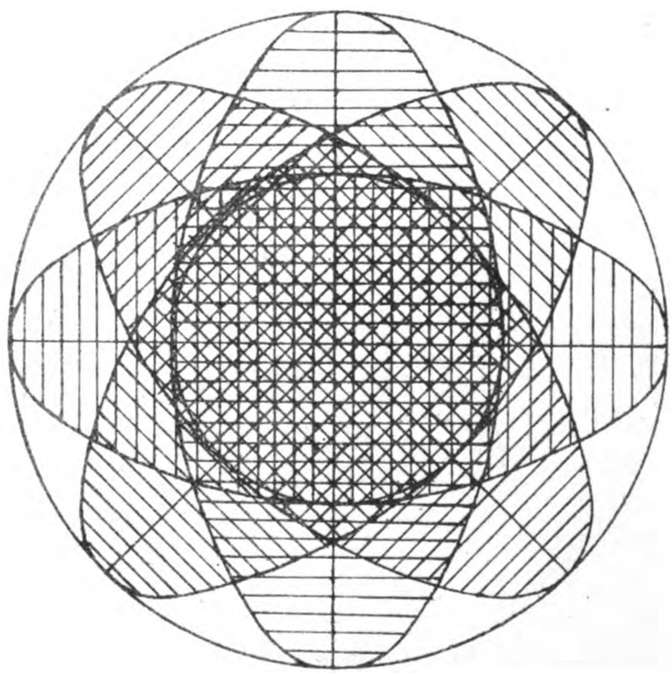


Fig. 27



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